Here again is Newton’s method for several variables.

\( \mathbf{x}^{(0)} \) initial guess (good we hope)

Repeat

Solve

\[
F'(\mathbf{x}^{(n)}) \mathbf{h} = -F(\mathbf{x}^{(n)})
\]

\[\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \mathbf{h}.\]

until \( \| \mathbf{h} \| \leq \epsilon_1 \) or \( \| F(\mathbf{x}^{(n)}) \| \leq \epsilon_2 \)

Example 1 Let

\[
F(\mathbf{x}) = \begin{pmatrix}
    x_1^2 + x_1x_2^3 - 9 \\
    3x_1^2x_2 - x_2^3 - 4
\end{pmatrix}.
\]

and let

\[\mathbf{x}^{(0)} = \begin{pmatrix}
    1.2 \\
    2.5
\end{pmatrix}.\]

We have that

\[
F'(\mathbf{x}) = \begin{pmatrix}
    2x_1 + x_2^3 & 3x_1x_2^2 \\
    6x_1x_2 & 3(x_1^2 - x_2^2)
\end{pmatrix}.
\]

Then

\[
F(\mathbf{x}^{(0)}) = \begin{pmatrix}
    11.1900 \\
    -8.8250
\end{pmatrix}
\]

\[
F'(\mathbf{x}^{(0)}) = \begin{pmatrix}
    18.0250 & 22.5000 \\
    18.0000 & -14.4300
\end{pmatrix}
\]

\[h = \begin{pmatrix}
    0.0558 \\
    -0.5420
\end{pmatrix}\]

Thus

\[\mathbf{x}^{(1)} = \begin{pmatrix}
    1.2558 \\
    1.9580
\end{pmatrix}.
\]

\[\mathbf{x}^{(2)} = \begin{pmatrix}
    1.3228 \\
    1.7728
\end{pmatrix}.
\]
\[ \mathbf{x}^{(3)} = \begin{pmatrix} 1.3362 \\ 1.7544 \end{pmatrix} . \]

\[ \mathbf{x}^{(4)} = \begin{pmatrix} 1.3364 \\ 1.7542 \end{pmatrix} . \]

The vector \( \mathbf{x}^{(4)} \) is correct to eight digit accuracy. The vector \( \mathbf{x}^{(5)} \) is correct to machine precision.

Under the correct (very complicated) conditions, convergence is quadratic. That is, if \( \mathbf{x}^* \) is the zero of \( \mathbf{F} \) that are looking for,

\[ \| \mathbf{x}^{(n+1)} - \mathbf{x}^* \| \leq C_F \| \mathbf{x}^{(n+1)} - \mathbf{x}^* \|^2 . \]

We now consider a special application of Newton’s method. Suppose we want to compute \( \sqrt{a} \) where \( a \) is a floating point number.

\[ a = d \cdot 2^e, \quad d \in [1/2, 1). \]

The cases where \( d \) is unnormalized or zero can be treated separately.

We rewrite \( a \) as

\[ a = \hat{d} \cdot 2^{2f}, \quad \hat{d} \in [1/4, 1) \]

where

\[
 e = \begin{cases} 
 2f & \text{e even} \\
 2f - 1 & \text{e odd} 
\end{cases}, \\
 \hat{d} = \begin{cases} 
 d & \text{e even} \\
 d/2 & \text{e odd} 
\end{cases}.
\]

Thus

\[ \sqrt{a} = \sqrt{\hat{d} \cdot 2^f}, \quad \sqrt{\hat{d}} \in [1/2, 1). \]

Use the fact that \( \hat{d} \) is the root of

\[ f(x) = x^2 - \hat{d} = 0. \]

To use Newton’s method, we note that

\[ f'(x) = 2x. \]

Thus as we showed earlier

\[ x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - (x_n^2 - \hat{d})/(2x_n) = 0.5(x_n + \hat{d}/x_n). \]

These iterates converge quadratically.
A reasonable $x_0$ is $3/4$. A slightly better $x_0$ is 

$$x_0 = \frac{1}{3}(1 + 2d),$$

which is the straight line between $(1/4, 1/2)$ and $(1, 1)$. This might save you one iteration.

With either of these initial guesses, Newton’s method converges to $\sqrt{d}$ in no more than 5 iterations.