A note on the error analysis of classical Gram–Schmidt

Alicja Smoktunowicz * Jesse L. Barlow †

October 2, 2005

Abstract

An error analysis result is given for classical Gram–Schmidt factorization of a full rank matrix $A$ into $A = QR$ where $Q$ is left orthogonal (has orthonormal columns) and $R$ is upper triangular. The work presented here shows that a similar bound in [Giraud at al, Numer. Math. 101(1):87–100,2005] is true, but only if the diagonals of $R$ are computed in a manner similar to Cholesky factorization of the normal equation matrix.

The classical Gram–Schmidt (CGS) orthogonal factorization is analyzed in a recent work of Giraud et al. [5] and in a number of other sources [3, 8, 11, 1, 4, 7], [10, §6.9], [2, §2.4.5].

For a matrix $A \in \mathbb{R}^{m \times n}$ ($m \geq n$) with rank$(A) = n$, in exact arithmetic, the algorithm produces a factorization

$$A = QR$$

(1)

where $Q$ is left orthogonal (i.e. $Q^T Q = I_n$), and $R \in \mathbb{R}^{n \times n}$ is upper triangular and nonsingular. In describing the algorithms, we use the notational conventions,

$$A = (a_1, \ldots, a_n), \quad Q = (q_1, \ldots, q_n),$$

$$R = (r_{jk}).$$

The algorithm forms $Q$ and $R$ from $A$ column by column as described in the following MATLAB-like pseudo-code.

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*Faculty of Mathematics and Information Science, Warsaw University of Technology, Pl. Politechniki 1, Warsaw, 00-661 Poland, e-mail: smok@mini.pw.edu.pl
†Department of Computer Science and Engineering, University Park, PA 16802-6822, USA, e-mail: barlow@cse.psu.edu. Jesse Barlow’s research was supported by NSF grant no. CCF-0429481.
Algorithm 1 (Classical Gram–Schmidt Orthogonal Factorization)

\[ r_{11} = \|a_1\|_2; q_1 = a_1 / r_{11}; \]
\[ R_1 = (r_{11}); Q_1 = (q_1); \]
\[ \text{for } k = 2 \text{ : } n \]
\[ s_k = Q_{k-1}^* a_k; \]
\[ v_k = a_k - Q_{k-1} s_k; \]
\[ r_{kk} = \|v_k\|_2; \]
\[ q_k = v_k / r_{kk}; \]
\[ R_k = \begin{pmatrix} \frac{k-1}{1} & 1 \\ 1 & \frac{k-1}{1} \end{pmatrix} \]
\[ Q_k = \begin{pmatrix} q_{k-1} \end{pmatrix} \]
\[ \text{end}; \]
\[ Q = Q_n; R = R_n; \]

As is well known [2, p.63,§2.4.3], in floating point arithmetic, \( Q \) is far from left orthogonal. The authors of [5] prove a number of results about classical Gram–Schmidt. This note shows that for one of their results (Lemma 1 in [5]), the diagonal elements \( r_{kk} \) should be computed differently from Algorithm 1, substituting a Cholesky-like formula for \( r_{kk} \) rather than setting \( r_{kk} = \|v_k\|_2 \). That change produces the next algorithm.

Algorithm 2 (Cholesky–like Classical Gram–Schmidt Orthogonal Factorization)

\[ r_{11} = \|a_1\|_2; q_1 = a_1 / r_{11}; \]
\[ R_1 = (r_{11}); Q_1 = (q_1); \]
\[ \text{for } k = 2 \text{ : } n \]
\[ s_k = Q_{k-1}^* a_k; \]
\[ v_k = a_k - Q_{k-1} s_k; \]
\[ \phi_k = ||s_k||_2; \]
\[ r_{kk} = (\psi_k - \phi_k)^{1/2} (\psi_k + \phi_k)^{1/2}; \]
\[ q_k = v_k / r_{kk}; \]
\[ R_k = \begin{pmatrix} \frac{k-1}{1} & 1 \\ 1 & \frac{k-1}{1} \end{pmatrix} \]
\[ Q_k = \begin{pmatrix} q_{k-1} \end{pmatrix} \]
\[ \text{end}; \]
\[ Q = Q_n; R = R_n; \]

This computation of \( r_{kk} \) does not affect the results in [5] on re-orthogonalization of \( q_k \) against \( Q_{k-1} \). Our numerical tests indicate that the main and quite useful result in [5, Theorem 2] still holds if \( r_{kk} \) is computed as in Algorithm 1 after the re-orthogonalization.

We assume that we are using a floating point arithmetic that satisfies the IEEE floating point standard. In IEEE arithmetic

\[ f(\ell(x + y)) = (x + y)(1 + \delta), \quad |\delta| \leq \epsilon_M \]
for results in the normalized range [9, p.32].

Letting $\varepsilon_M$ be the machine unit, we follow Golub and Van Loan [6, §2.4.6], we use the linear approximation

$$(1 + \varepsilon_M)^{p(n)} = 1 + p(n)\varepsilon_M + O(\varepsilon_M^2)$$

for a modest function $p(n)$, and assume that the $O(\varepsilon_M^2)$ does not make a significant contribution.

We give a slightly different version of Lemma 1 from [5] for Algorithm 2. After defining the four functions

$$c_1(m, k) = \begin{cases} 1 & k = 1 \\ 2\sqrt{m}k + 2\sqrt{k} & k = 2, \ldots, n, \end{cases}$$

$$c_2(m, k) = \begin{cases} m + 2 & k = 1 \\ 3.5mk^2 - 1.5mk + 16k & k = 2, \ldots, n, \end{cases}$$

$$c_3(m, k) = 0.5c_2(m, k), \quad c_4(m, k) = c_2(m, k) + 2c_1(m, k),$$

the new version of the result in [5, Lemma 1] is Theorem 1.

**Theorem 1** Assume that in floating point arithmetic with machine unit $\varepsilon_M$, the $R$ resulting from Algorithm 2, we have

$$c_4(m, n)\varepsilon_M ||R||_2^2 ||R^{-1}||_2^2 < 1.\quad (3)$$

Then, for $k = 1, \ldots, n$ the computed matrices $R_k$ and $Q_k$ satisfy

$$A_k + \Delta A_k = Q_k R_k, \quad ||\Delta A||_2 \leq c_1(m, k)||A_k||_2 \varepsilon_M + O(\varepsilon_M^2), \quad (4)$$

$$R_k^T R_k = A_k^T A_k + E_k, \quad ||E_k||_2 \leq c_2(m, k)||A_k||_2^2 \varepsilon_M + O(\varepsilon_M^2), \quad (5)$$

$$||R_k||_2 = ||A_k||_2 (1 + \mu_k), \quad |\mu_k| \leq c_3(m, k)\varepsilon_M + O(\varepsilon_M^2), \quad (6)$$

$$||I - Q_k^T Q_k||_2 \leq c_4(m, k)||R_k||_2^2 ||R_k^{-1}||_2^2 \varepsilon_M + O(\varepsilon_M^2), \quad (7)$$

$$||Q_k||_2 \leq \sqrt{2} + O(\varepsilon_M^2). \quad (8)$$

In exact arithmetic, $||R_k||_2 = ||A_k||_2$ and $||R_k^{-1}||_2 = ||A_k^{-1}||_2$. Equation (5) states that, in floating point arithmetic, the first equality can be replaced by $||R_k||_2 = ||A_k||_2 + O(\varepsilon_M)$. The latter inequality is not true in floating point arithmetic.

The proof is given in the appendix. To show that the conclusion of Theorem 1 does not hold for Algorithm 1, we give the following example.

**Example 1** We produced a $6 \times 6$ matrix with the following MATLAB code.

```matlab
B=hilb(6);
A1=ones(6,3)+B(:,1:3)*1e-2;
B=pascal(6);
A2=B(:,1:3);
A=[A1 A2];
```
Table 1: Orthogonality and Normal Equations Error from CGS Algorithms

| Algorithm | \(|A^T A - R^T R|\|_2/\|A\|_2^2| | \|I - Q^T Q|\|_2 | |
|-----------|-----------------|-----------------|
| Algorithm 1 | 4.7369e-08 | 8.9471e-04 |
| Algorithm 2 | 7.2912e-17 | 5.8228e-03 |

The command \texttt{hilb(6)} produces the 6\times6 Hilbert matrix, the command \texttt{ones(6,3)} produces a 6 \times 3 matrix of ones, and the command \texttt{pascal(6)} produces a 6 \times 6 matrix from Pascal’s triangle. The condition number of \(R\) from Algorithm 2, \(\kappa_2(R) = \|R\|_2\|R^{-1}\|_2\), computed by the MATLAB command \texttt{cond}, is 8.9563 \cdot 10^7, thus given that \(\varepsilon_M \approx 2.2206 \cdot 10^{-16}\) in IEEE double precision, \(R\) is neither well-conditioned nor near singular.

We computed the \(Q\)-\(R\) factorization using Algorithm 1 and then we computed using Algorithm 2. The resulting \(Q\) and \(R\) satisfy the results in Table 1.

The bound on \(\|A^T A - R^T R\|_2\) in (5) appears to be satisfied if \(r_{kk}\) is computed as in Algorithm 2, but it is not if \(r_{kk}\) is computed as in Algorithm 1. For both methods of computing \(r_{kk}\), the orthogonality error in \(Q\) is about what would be expected from [5, Theorem 1].

Conclusion

The upper triangular factor \(R\) from classical Gram-Schmidt has been shown to satisfy the bound (??) provided that the diagonal elements of \(R\) are computed as they are in the Cholesky factorization of the normal equations matrix. If these diagonal elements as in standard versions of classical Gram-Schmidt, no bound such as (5) may be guaranteed.

References


REFERENCES


Appendix. Proof of Theorem 1

To set up the proof of Theorem 1, we require a lemma.

Lemma 1 Let $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$ be the results of Algorithm 2 in floating point arithmetic with machine unit $\varepsilon_M$ and that $R$ satisfies (3). Then

$$r_{11} = \|a_1\|_2(1 + \delta_1), \quad |\delta_1| \leq (0.5m + 1)\varepsilon_M + O(\varepsilon_M^2) \quad (9)$$

and for $k = 2, \ldots, n$

$$r_{kk} = \left(\|a_k\|_2^2(1 + \delta_k) - \|s_k\|_2^2(1 + \Delta_k)\right)^{1/2}, \quad (10)$$

$$|\delta_k|, |\Delta_k| \leq (m + 8)\varepsilon_M + O(\varepsilon_M^2),$$

$$\|s_k\|_2 \leq \|a_k\|_2(1 + \zeta), \quad |\zeta| \leq (m + 2)\varepsilon_M + O(\varepsilon_M^3). \quad (11)$$

Proof. Equation (9) is just the error in the computation of $\|a_1\|_2$. In the computation of $r_{kk}, k = 2, \ldots, n$, note that

$$\psi_k = f\ell(\|a_k\|_2) = \|a_k\|_2(1 + \epsilon_i^k), \quad (12)$$

$$\phi_k = f\ell(\|s_k\|_2) = \|s_k\|_2(1 + \epsilon_i^k), \quad (13)$$

$$|\epsilon_i^k| \leq (0.5m + 1)\varepsilon_M + O(\varepsilon_M^2), \quad i = 1, 2.$$

Using (3), we conclude that $R$ is nonsingular, thus $r_{kk} > 0$ for all $k$. Thus in Algorithm 2, $r_{kk} > 0$ only if $\psi_k > \phi_k$. 


To get (10), note that
\[ r_{kk} = \sqrt{\psi_k - \phi_k \sqrt{\psi_k}} + \phi_k (1 + \varepsilon^{(k)})_3, \quad |\varepsilon^{(k)}_3| \leq 3\varepsilon_M + O(\varepsilon^2_M). \]
Thus using (12) and (13), we have
\[
\begin{align*}
    r_{kk} &= \sqrt{\|a_k\|^2_2 (1 + \varepsilon^{(k)}_1)^2 - \|s_k\|^2_2 (1 + \varepsilon^{(k)}_3)^2 (1 + \varepsilon^{(k)}_3)} \\
    &= \left(\|a_k\|^2_2 (1 + \delta_k) - \|s_k\|^2_2 (1 + \Delta_k)\right)^{1/2}
\end{align*}
\]
where
\[
\begin{align*}
    \delta_k &= (1 + \varepsilon^{(k)}_1)^2 (1 + \varepsilon^{(k)}_3)^2 - 1, \\
    \Delta_k &= (1 + \varepsilon^{(k)}_2)^2 (1 + \varepsilon^{(k)}_3)^2 - 1.
\end{align*}
\]
That yields
\[ |\delta_k|, |\Delta_k| \leq (m + 8)\varepsilon_M + O(\varepsilon^2_M). \]
Therefore \( r_{kk} \) satisfies (10).

Since \( \psi_k > \phi_k \) as outlined above, from (12)–(13), we have
\[ \psi_k = \|a_k\|_2 (1 + \varepsilon^{(k)}_1) > \phi_k = \|s_k\|_2 (1 + \varepsilon^{(k)}_2) \]
thus
\[ \|s_k\|_2 < \|a_k\|_2 (1 + \varepsilon^{(k)}_1)(1 + \varepsilon^{(k)}_2)^{-1} \leq \|a_k\|_2 (1 + \zeta) \]
where \( \zeta \) satisfies (11). \( \square \)

As a consequence of the singular value version of the Cauchy interlace theorem [6, p.449-450, Corollary 8.6.3], we have that \( \|R_k\|_2 \leq \|R\|_2 \) and \( \|R^{-1}_k\|_2 \leq \|R^{-1}\|_2 \). We will use these facts freely in the proof of Theorem 1.

We can now prove Theorem 1.

**Proof.** [of Theorem 1] The results (4)–(5) are proven by induction on \( k \). First, consider \( k = 1 \). From Lemma 1, we have (9), so
\[ r_{11} = \|a_1\|_2 (1 + \delta_1), \quad |\delta_1| \leq (0.5m + 1)\varepsilon_M + O(\varepsilon^2_M) \]
which implies that
\[
\begin{align*}
    R^T_1 R_1 &= r_{11}^2 = \|a_1\|_2^2 (1 + \delta_1)^2 \\
    &= A_1^T A_1 (1 + \delta_1)^2 = A_1^T A_1 + E_1
\end{align*}
\]
where
\[ E_1 = 2\delta_1 A_1^T A_1 + \delta_1^2 A_1^T A_1. \]
Thus
\[ \|E_1\|_2 = \|E_1\|_2 \leq (m + 2)\|a_1\|_2^2 \varepsilon_M + O(\varepsilon^2_M) = (m + 2)\|A_1\|_2^2 \varepsilon_M + O(\varepsilon^2_M). \]
Also, we can conclude from standard error bounds that 
\[ q_1 = (I + G_1) a_1 / r_{11}, \quad \| G_1 \|_2 \leq \varepsilon_M. \]
Therefore \[ A_1 - Q_1 R_1 = a_1 - q_1 r_{11} = -G_1 a_1 \]
so that 
\[ \| A_1 - Q_1 R_1 \|_2 = \| a_1 - q_1 r_{11} \|_2 \leq \| G_1 \|_2 \| a_1 \|_2 \leq \varepsilon_M \| a_1 \|_2. \]  \hspace{1cm} (14)

Assume that (4)-(8) hold for \( k - 1 \), and prove them for \( k \). We first prove (4)-(5), and then show that (6)-(8) follow.

First, we start with error bounds of the computation of the vectors \( s_k v_k \), and \( q_k \) to prove (4). Note that 
\[ s_k = f(\ell(Q_{k-1}^T a_k) = Q_{k-1}^T a_k - \delta s_k \]  \hspace{1cm} (15)
where 
\[ \| \delta s_k \|_2 \leq m \sqrt{k-1} \| Q_{k-1} \|_2 \| a_k \|_2 \varepsilon_M + O(\varepsilon_M^2) \]
\[ \leq \sqrt{2(k-1)} m \| a_k \|_2 \varepsilon_M + O(\varepsilon_M^2). \]  \hspace{1cm} (16)

Also, we have 
\[ v_k = f(\ell(a_k - Q_{k-1} s_k) = a_k - Q_{k-1} s_k - \delta v_k \]  \hspace{1cm} (17)
where 
\[ \| \delta v_k \|_2 \leq \| a_k \|_2 \varepsilon_M + \sqrt{k-1} m \| Q_{k-1} \|_2 \| s_k \|_2 \varepsilon_M + O(\varepsilon_M^2). \]

From (11), the bound on \( \| s_k \|_2 \) in (11), and the induction hypothesis on \( Q_{k-1} \), we have 
\[ \| \delta v_k \|_2 \leq (\sqrt{2(k-1)} m + 1) \| a_k \|_2 \varepsilon_M + O(\varepsilon_M^2). \]  \hspace{1cm} (18)

Again using the bound on \( \| s_k \|_2 \) in (11), we note that 
\[ \| v_k + \delta v_k \|_2^2 = \| a_k \|_2^2 - 2 a_k^T Q_{k-1} s_k + \| Q_{k-1} s_k \|_2^2 \]
\[ = \| a_k \|_2^2 - 2 \| s_k \|_2^2 + \| Q_{k-1} s_k \|_2^2 - 2 (\delta s_k)^T s_k \]
\[ = \| a_k \|_2^2 - 2 \| s_k \|_2^2 + \| Q_{k-1} s_k \|_2^2 - 2 (\delta s_k)^T s_k \]
\[ \leq \| a_k \|_2^2 - 2 \| s_k \|_2^2 + 2 \| s_k \|_2^2 - 2 (\delta s_k)^T s_k \]
\[ = \| a_k \|_2^2 - 2 (\delta s_k)^T s_k \]
\[ \leq \| a_k \|_2^2 + 2 \| \delta s_k \|_2 \| s_k \|_2 \]
\[ = \| a_k \|_2^2 + 2 \| \delta s_k \|_2 \| a_k \|_2 + O(\varepsilon_M^2) \]
\[ \leq \| a_k \|_2^2 (1 + \sqrt{2(k-1)} m \varepsilon_M)^2 + O(\varepsilon_M^2). \]

Thus 
\[ \| v_k \|_2 \leq \| a_k \|_2 (1 + (3 \sqrt{2(k-1)} m) \varepsilon_M) + O(\varepsilon_M^2) = \| a_k \|_2 + O(\varepsilon_M). \]
We note that
\[ q_k = (I + G_k)v_k/r_{kk}, \quad ||G_k||_2 \leq \varepsilon_M. \]
If we let
\[ \Delta A_k = Q_k R_k - A_k \]
then
\[ \Delta A_k = (\Delta A_{k-1} \quad \delta a_k) \]
where
\[ \delta a_k = (I + G_k)v_k + Q_{k-1}s_k - a_k, \]
\[ = G_k v_k - \delta v_k. \]
That yields
\[ ||\delta a_k||_2 \leq ||G_k||_2||v_k||_2 + ||\delta v_k||_2 \leq (2\sqrt{2(k-1)m} + 2\varepsilon_M||a_k||_2 + O(\varepsilon_M^3). \]
To bound \( ||\Delta A_k||_F \), we give a recurrence for bounding \( ||\Delta A_k||_F \) in terms of \( ||A_k||_F \), then use the bound \( ||A_k||_F \leq \sqrt{k}||A_k||_2 \). We show that
\[ ||\Delta A_k||_F \leq \hat{c}_1(m, k)||A_k||_F \varepsilon_M + O(\varepsilon_M^3). \]
For \( k = 1 \),
\[ ||\Delta A_1||_F = ||a_1||_2 = \varepsilon_M||a_1||_2 = \varepsilon_M||A_1||_F. \]
Using properties of the Frobenius norm,
\[ ||\Delta A_k||_F^2 \leq ||\Delta A_{k-1}||_F^2 + ||\delta a_k||_2^2 \]
\[ \leq \hat{c}_1^2(m, k-1)||A_{k-1}||_F^2 + (2\sqrt{2(k-1)m} + 2\varepsilon_M||a_k||_2^2)\varepsilon_M^2 + O(\varepsilon_M^3) \]
\[ \leq \max\{\hat{c}_1^2(m, k-1), (2\sqrt{2(k-1)m} + 2\varepsilon_M||a_k||_2^2)\varepsilon_M^2 + O(\varepsilon_M^3) \}
\[ = \hat{c}_1^2(m, k)||A_k||_F^2 \varepsilon_M^2 + O(\varepsilon_M^3). \quad (19) \]
A quick induction argument yields
\[ \hat{c}_1(m, k) = 2\sqrt{2(k-1)m} + 2 \leq 2\sqrt{2km} + 2. \]
Thus
\[ ||\Delta A_k||_2 \leq ||\Delta A_k||_F \leq \hat{c}_1(m, k)\varepsilon_M||A_k||_F + O(\varepsilon_M^3) \leq \sqrt{k}\hat{c}_1(m, k)||A_k||_2 + O(\varepsilon_M^3) \]
yielding (4) with \( c_1(m, k) = 2\sqrt{2mk} + 2\sqrt{k} \geq \sqrt{k}\hat{c}_1(m, k). \)
To prove (5), note that
\[ E_k = R_k^T R_k - A_k^T A_k = \begin{pmatrix} k-1 & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} E_{k-1} & w_k \\ w_k^T & \delta_{kk} \end{pmatrix} \]
where using Lemma 1, we have
\[
\begin{align*}
w_k &= R_{k-1}^T s_k - A_{k-1}^T a_k, \\
e_{kk} &= s_k^T s_k + r_{kk}^2 - a_k^T a_k \\
&= \delta_s a_k^T a_k - \Delta s_k^T s_k.
\end{align*}
\]

Using the bounds on \(\delta_s\) and \(\Delta_s\) in (10), we have
\[
|e_{kk}| \leq |\delta_s||a_k||s_k|^2 + |\Delta_s||s_k||^2
\leq (|\delta_s| + |\Delta_s|)||a_k||^2 + O(\varepsilon_M^2)
\leq 2(m + 8)||a_k||^2 + O(\varepsilon_M^2)
\leq 2(m + 8)||A_{k-1}||^2 + O(\varepsilon_M^2).
\]

Since
\[
s_k + \delta s_k = Q_{k-1}^T a_k, \quad A_{k-1} + \Delta A_{k-1} = Q_{k-1}^T R_{k-1}
\]
we have
\[
\begin{align*}
w_k &= R_{k-1}^T s_k - A_{k-1}^T a_k \\
&= R_{k-1}^T Q_{k-1}^T a_k - R_{k-1}^T \delta s_k - A_{k-1}^T a_k \\
&= \Delta A_{k-1}^T a_k - R_{k-1}^T \delta s_k.
\end{align*}
\]

So that \(\|w_k\|\) has the bound
\[
\begin{align*}
\|w_k\| \leq \|\Delta A_{k-1}\|_2||a_k||_2 + \|R_{k-1}\|_2||\delta s_k||_2 + O(\varepsilon_M^2)
\leq (c_1(m, k - 1)||A_{k-1}\|_2||a_k||_2 + \sqrt{2(k - 1)m}||A_{k-1}\|_2||a_k||_2)\varepsilon_M
\leq [2\sqrt{2m(k - 1)} + 2\sqrt{k - 1} + \sqrt{2(k - 1)m}]||A_{k-1}\|_2||a_k||_2\varepsilon_M + O(\varepsilon_M^2)
\leq 7m(k - 1)||A_k||^2\varepsilon_M + O(\varepsilon_M^2)
\end{align*}
\]

We have that
\[
\begin{align*}
\|E_k\| \leq \|E_{k-1}\|_2 + \|e_{kk}\|_2 + \|w_k\|_2
\leq \max\{|E_{k-1}|_2, |e_{kk}|\} + \|w_k\|_2
\leq [\max\{c_2(m, k - 1), 2(m + 8)\} + 7m(k - 1)]||A_k||^2\varepsilon_M + O(\varepsilon_M^2)
\leq [c_2(m, k - 1) + 2(m + 8) + 7m(k - 1)]||A_k||^2\varepsilon_M + O(\varepsilon_M^2)
\leq c_2(m, k)||A_k||^2\varepsilon_M + O(\varepsilon_M^2)
\end{align*}
\]

where
\[
c_2(m, k) = \sum_{j=1}^{k}[2(m + 8) + 7m(j - 1)]
= 3.5m(k - 1)k + 2mk + 16k.
\]

Thus we have the expression for \(c_2(m, k)\) given in equation 2.
To prove (6)-(8), we simply apply (4)-(5). Equation (6) results from noting that
\[
\|R_k\|_2^2 = \|R_k^T R_k\|_2 = \|A_k^T A_k + E_k\|_2 \\
\leq \|A_k^T A_k\|_2 + \|E_k\|_2 \leq (1 + c_2(m, k)\epsilon_M)\|A_k\|_2^2 + O(\epsilon_M^2).
\]
Thus,
\[
\|R_k\|_2 \leq (1 + c_3(m, k)\epsilon_M)\|A_k\|_2 + O(\epsilon_M^2)
\]
where
\[
1 + c_3(m, k)\epsilon_M + O(\epsilon_M^2) = \sqrt{1 + c_2(m, k)},
\]
that is, \(c_3(m, k) = 0.5c_2(m, k)\). Reversing the roles of \(R_k\) and \(A_k\) yields
\[
\|A_k\|_2 \leq (1 + c_3(m, k)\epsilon_M)\|R_k\|_2 + O(\epsilon_M^2),
\]
thus we have (6).

To get (7), we note that
\[
Q_k = (A_k + \Delta A_k)R_k^{-1}
\]
so that
\[
I - Q_k^T Q_k = R_k^{-T}(R_k^T R_k - (A_k + \Delta A_k)^T (A_k + \Delta A_k))R_k^{-1} \\
= R_k^{-T}(E_k - A_k^T \Delta A_k - (\Delta A_k)^T A_k - \Delta A_k)^T \Delta A_k)R_k^{-1}.
\]
Thus
\[
\|I - Q_k^T Q_k\|_2 \leq \|R_k^{-1}\|_2^2 (\|E_k\|_2 + 2\|\Delta A_k\|_2\|A_k\|_2 + \|A_k\|_2^2) \\
\leq \|R_k^{-1}\|_2^2 (c_2(m, k)\|A_k\|_2^2 + 2c_1(m, k)\|A_k\|_2 + \epsilon_M^2)\|A_k\|_2^2 + \epsilon_M + O(\epsilon_M^2) \\
\leq \|R_k\|_2^2 (c_2(m, k) + 2c_1(m, k))\epsilon_M + O(\epsilon_M^2) \\
= c_4(m, k)\|R_k\|_2^2 + O(\epsilon_M^2)
\]
where \(c_4(m, k) = c_2(m, k) + 2c_1(m, k)\).

Finally, to get (8), we have that
\[
\|Q_k\|_2^2 = \|Q_k^T Q_k\|_2 = \|I - Q_k^T Q_k - I\|_2 \\
\leq \|I\|_2 + \|I - Q_k^T Q_k\|_2 \\
\leq 1 + \|I - Q_k^T Q_k\|_2 \\
\leq 1 + c_4(m, k)\|R_k\|_2^2 (R_k^{-1}\|A_k\|_2^2 + \epsilon_M + O(\epsilon_M^2) \leq 2 + O(\epsilon_M^2).
\]
Taking square roots yields (8). \(\square\)