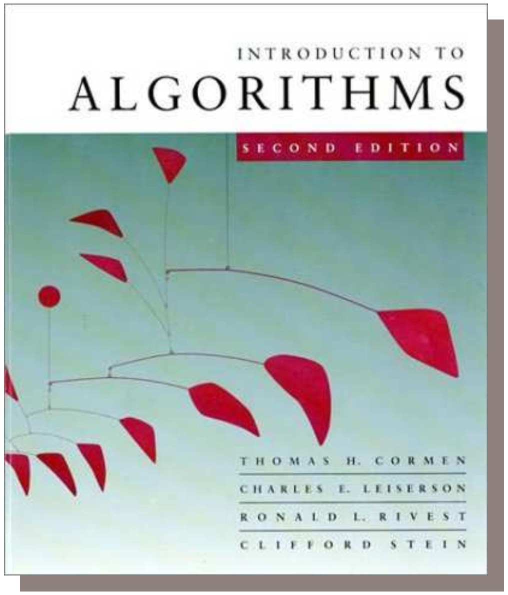
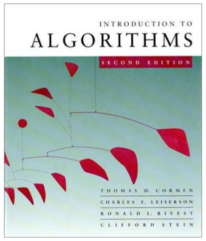
Data Structures and Algorithms
CSE 465



LECTURE 9

- **Finishing QuickSort**
- **Randomized Selection**

Sofya Raskhodnikova and Adam Smith



Reminder: QuickSort

Quicksort an n-element array:

1. ***Divide:*** Partition the array around a ***pivot*** x such that elements on left are $\leq x$ and elements on right are $\geq x$

2. ***Conquer:*** Recursively sort the tw
3. ***Combine:*** Nothing!

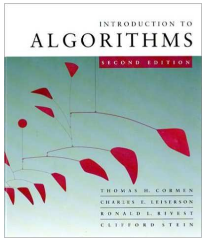
Last lecture: use *random* element $A[p]$ as pivot

Key: *Linear-time partitioning subroutine.*

Review

Recall: A split is "OK" when both pieces have $\geq \lfloor n/4 \rfloor$ elements

- What is the probability of an OK split with a random pivot? (Answer ≈ ½)
- Suppose we keep trying pivots and partitioning until we find an OK split.
  - What is the **expected** number of pivots we try before finding a good one? (Answer ≈ 2)
  - What is the expected running time it takes to find a good pivot this way? (Answer: $\Theta(n)$)



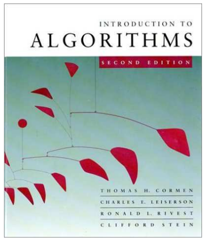
Randomized quicksort analysis

Let $T(n)$ = the **random variable** for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.



Analysis (continued)

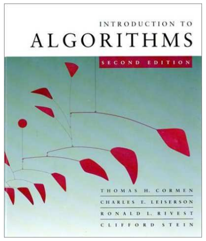
$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)$$

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Take expectations of both sides.

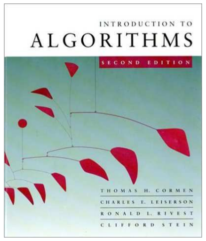


Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

Linearity of expectation.



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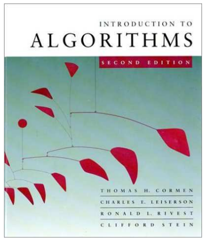
$$= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

Independence of X_k from other random choices.



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

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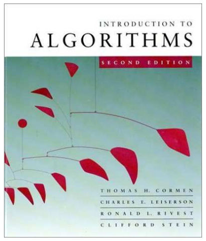
$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.



Calculating expectation

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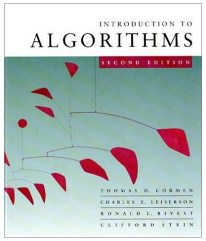
$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.



A recurrence on expected value

- We want to know the asymptotc behaviour of

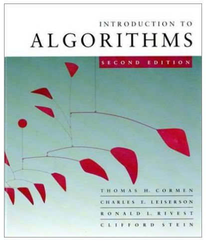
$$F(n) = E[T(n)]$$

real number

random variable

- We know it satisfies

$$F(n) = \frac{2}{n} \sum_{k=2}^{n-1} F(k) + \Theta(n)$$



Hairy recurrence

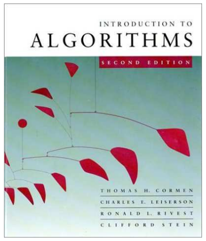
$$F(n) = \frac{2}{n}\sum_{k=2}^{n-1} F(k) + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $F(n) \le a n \lg n$ for constant $a > 0$.

- Choose a large enough so that $a n \lg n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

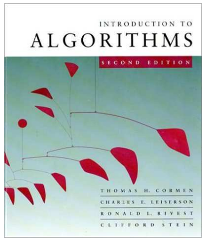
Use fact: $\sum_{k=2}^{n-1} k \lg k \le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).



Substitution method

$$F(n) \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitute inductive hypothesis.

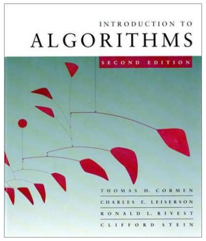


Substitution method

$$F(n) \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.



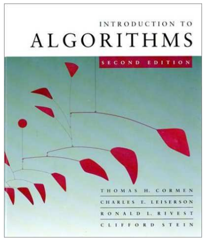
Substitution method

$$F(n) \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired – residual***.



Substitution method

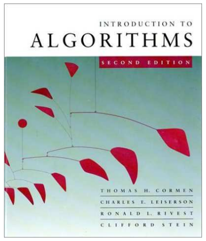
$$F(n) \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$= \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

$$\le an\lg n$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.



Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.

Order statistics

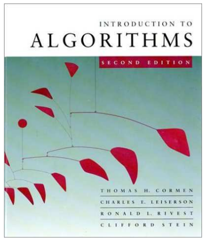
Select the ith smallest of n elements (the element with ***rank*** i).

- $i = 1$: ***minimum***;
- $i = n$: ***maximum***;
- $i = \lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$: ***median***.

Naive algorithm: Sort and index ith element.

Worst-case running time $= \Theta(n \lg n) + \Theta(1)$
$= \Theta(n \lg n)$,

using merge sort (*not* quicksort…).



Background: Properties of the Expectation

Here are some useful properties of the expectation:

- For any random variables *A*, *B*, and constants *c*,*d*:

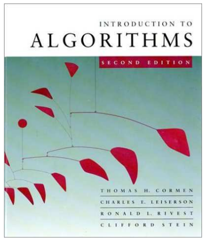
$$E[cA+dB] = c\,E[\mathrm{A}] + d\,E[B]$$

- For any two **independent** random variables

$$E[AB] = E[A]\,E[B]$$

- If *A* only takes the values 0 and 1 then

$$E[A] = \mathrm{Prob}[\,A=1\,]$$



Background: Geometric Random Variables

- Suppose we flip a coin many times independently, and each time the probability that it comes up "heads" is p
  - If X is the number of times we flip the coin before getting heads, then X is a geometric random variable with parameter p
- *E.g.*
  - "How many times do we roll a die until we see a 1"? (Geometric with p =1/6.)
  - "How many times do we flip a fair (50:50) coin before getting heads?" (Geometric with $p = ½$.)
  - "How many pivots do we choose before seeing an OK split?" (Geometric with $p = ½$.)
- For geometric r.v. with parameter p, $\boxed{E(X) = 1/p}$