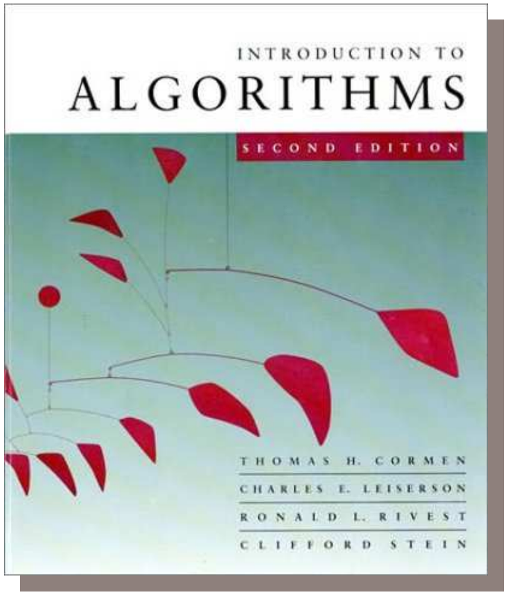
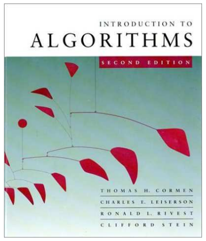
Data Structures and Algorithms
CSE 465



LECTURE 6
Solving Recurrences
- Master Theorem

Sofya Raskhodnikova and Adam Smith



Review questions

- Guess the solution to the recurrence: $T(n)=8T(n/2)+cn$.

(Answer: $\Theta(n^3)$.)

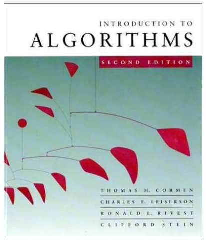
- Draw the recursion tree for this recurrence.

  a. What is its height?

(Answer: $h=\log n$.)

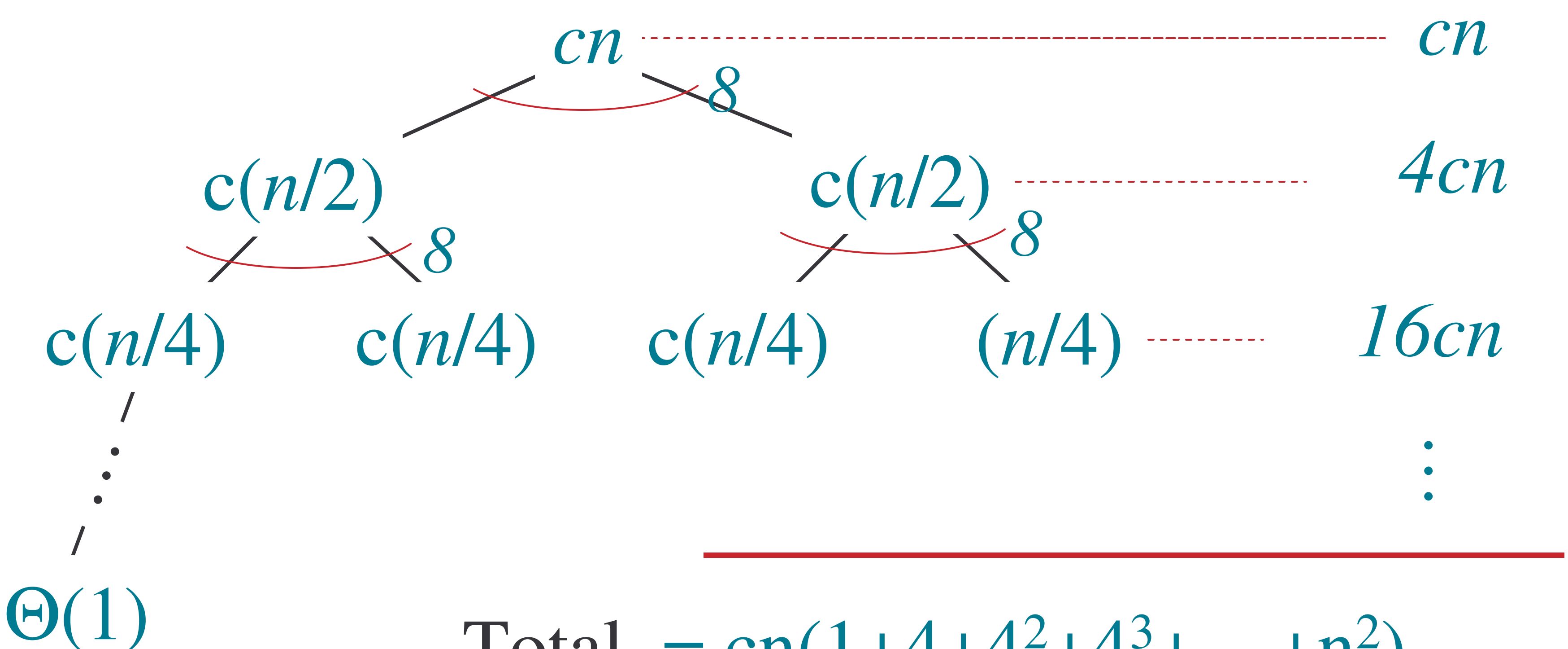
  b. What is the number of leaves in the tree?

(Answer: $8^h = 8^{\log n} = n^{\log 8} = n^3$.)



Review questions: recursion tree

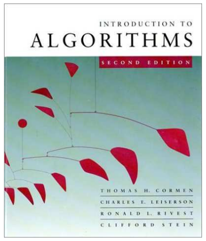
Solve $T(n)=8T(n/2)+cn$:



Total $= cn(1+4+4^2+4^3+\ldots+n^2)$

$= \Theta(n^3)$ *geometric series*

S. Raskhodnikova and A. Smith. Based on notes by E. Demaine and C. Leiserson

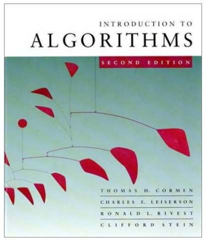


The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive, that is $f(n) > 0$ for all $n > n_0$.

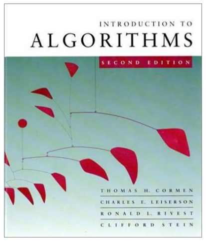


Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.



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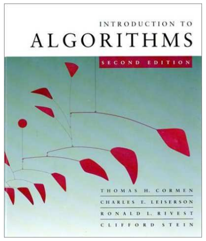
   Solution: $T(n) = \Theta(n^{\log_b a})$.

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.



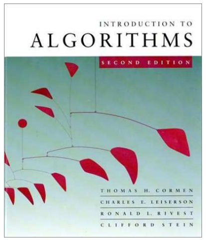
Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

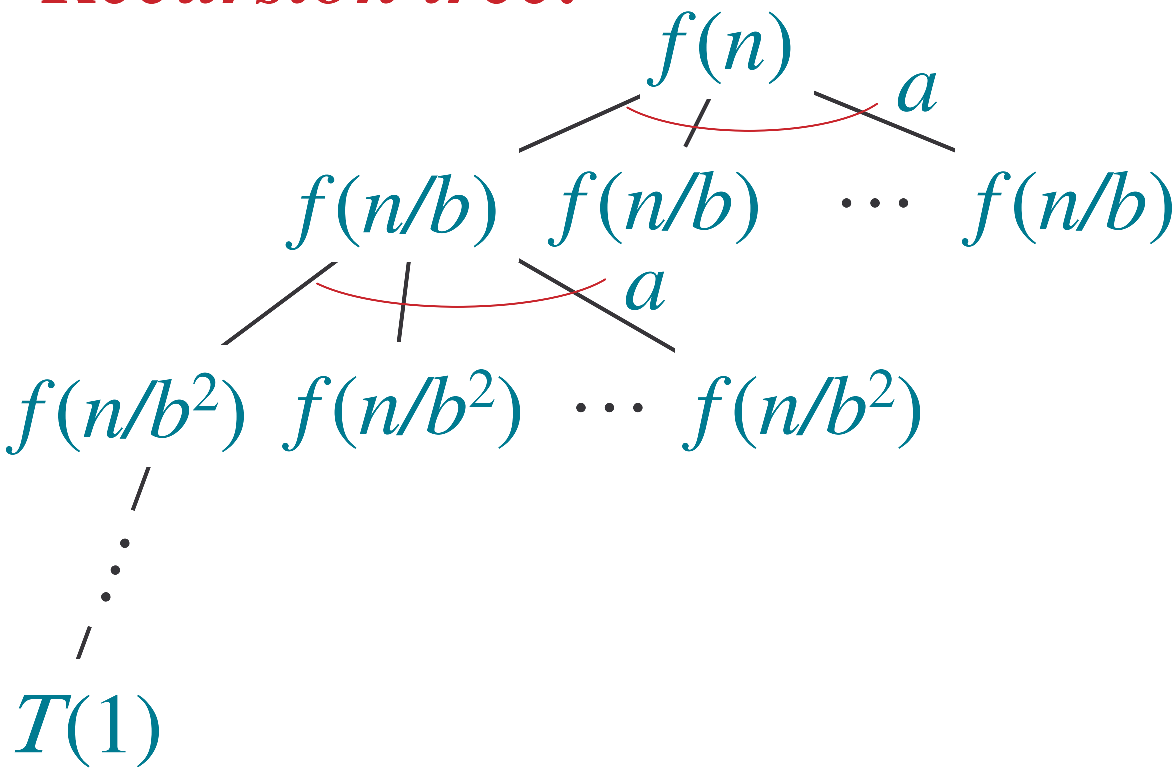
   and $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \le c f(n)$ for some constant $c < 1$.

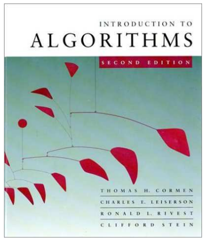
   Solution: $T(n) = \Theta(f(n))$.



Idea of master theorem

Recursion tree:





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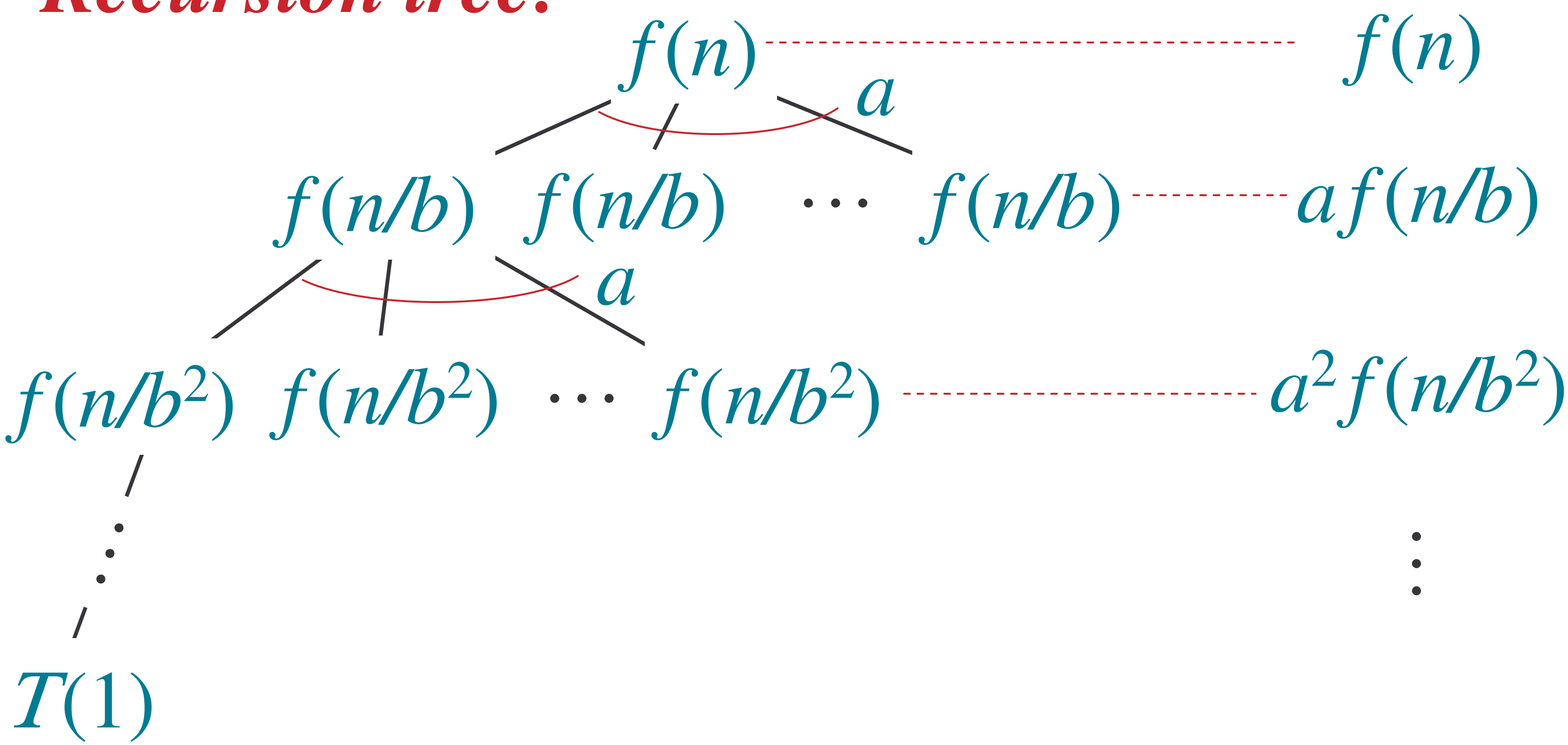

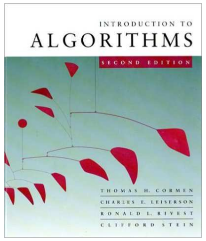

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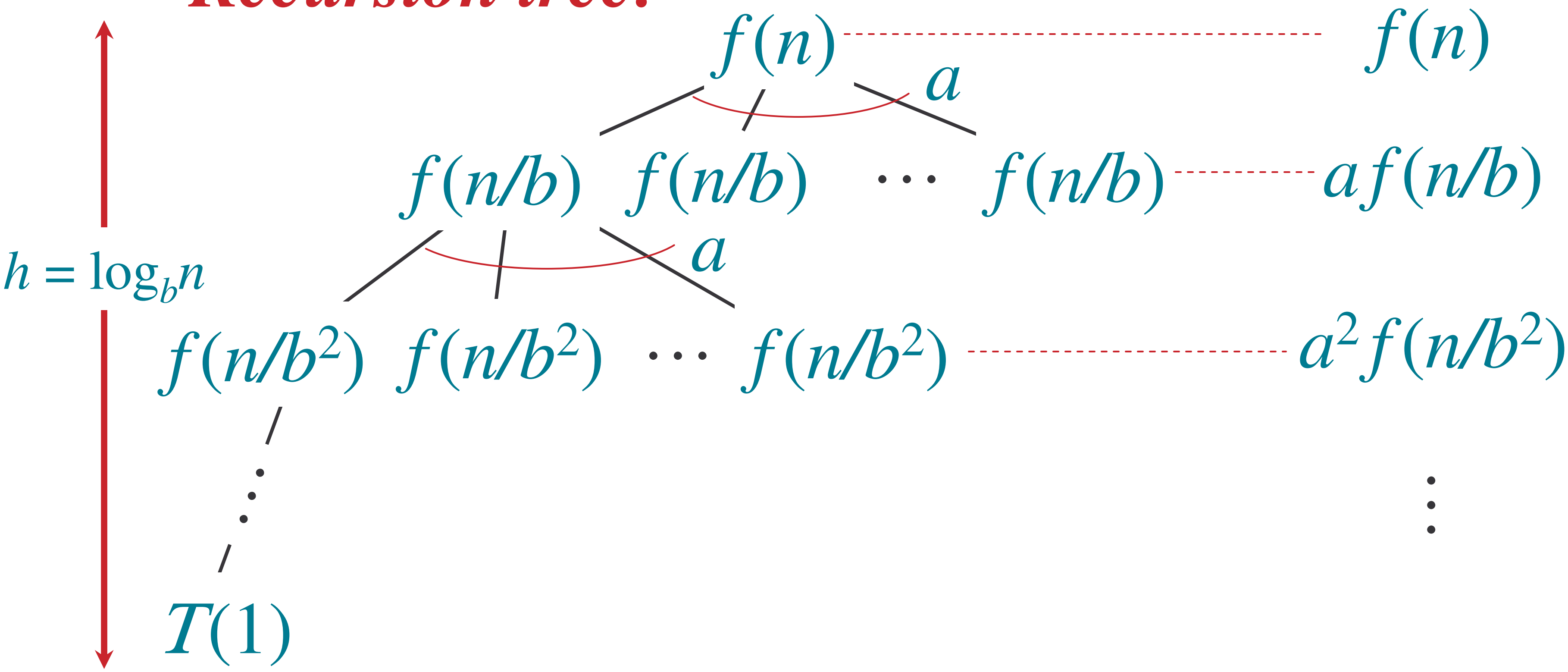

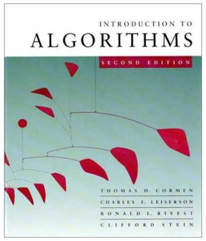

Idea of master theorem

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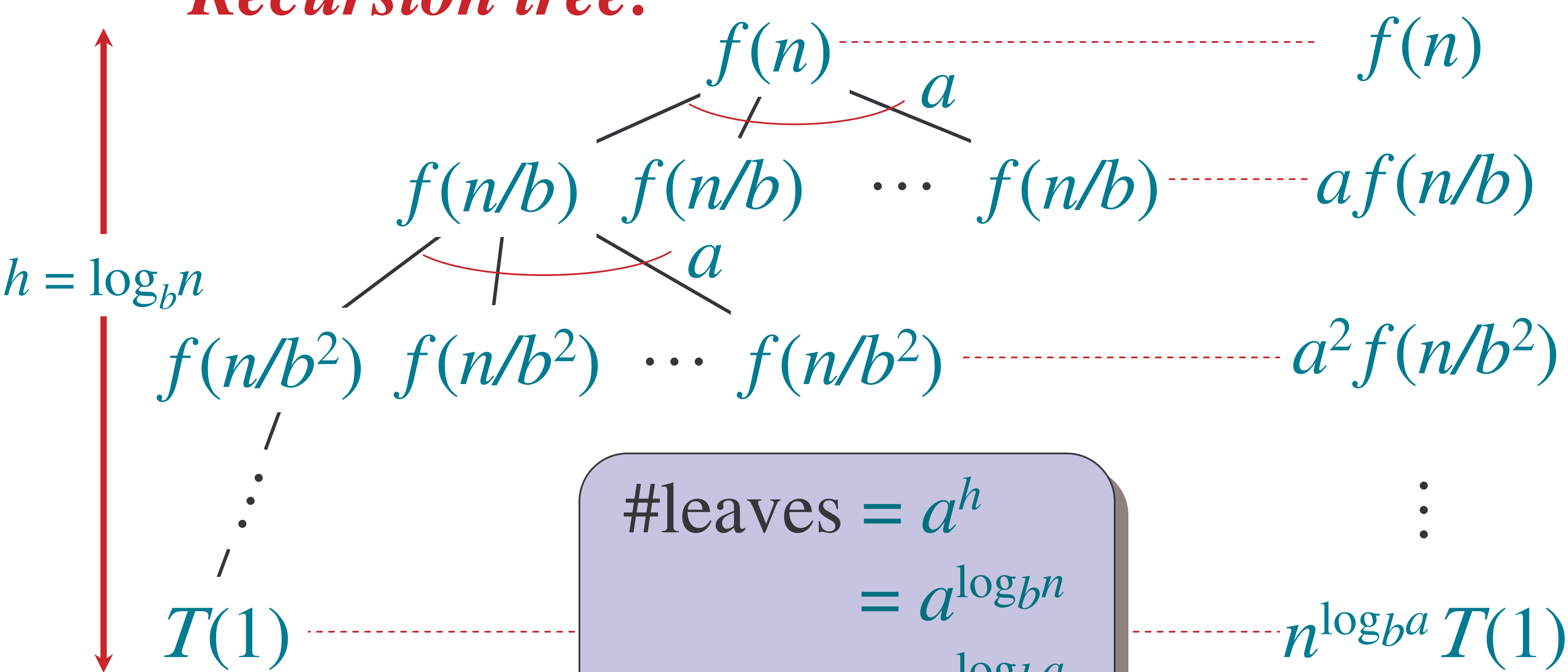
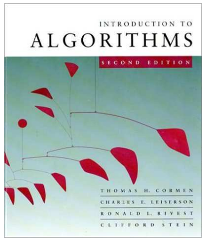


$$\#\text{leaves} = a^h = a^{\log_b n} = n^{\log_b a}$$



Idea of master theorem

Recursion tree:

$f(n)$ .... $f(n)$

a

$f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b)$ .... $a f(n/b)$

a

$h = \log_b n$

$f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2)$ .... $a^2 f(n/b^2)$

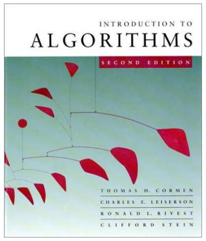
$\vdots$

$T(1)$ .... $n^{\log_b a} T(1)$

$\Theta(n^{\log_b a})$

CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

S. Raskhodnikova and A. Smith. Based on notes by E. Demaine and C. Leiserson



Idea of master theorem

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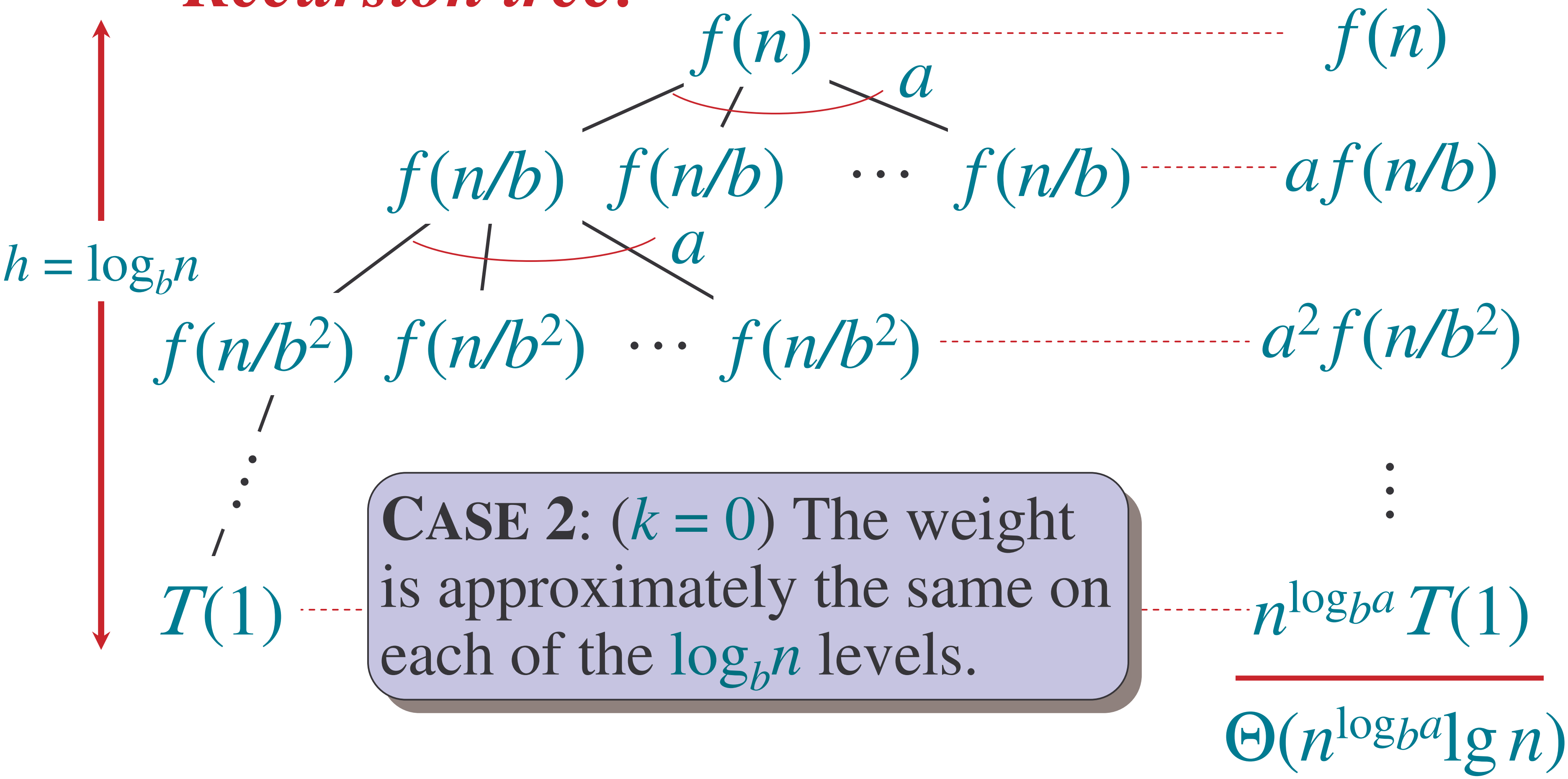

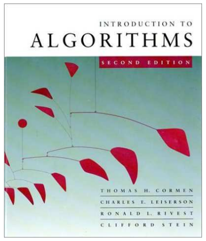

Idea of master theorem

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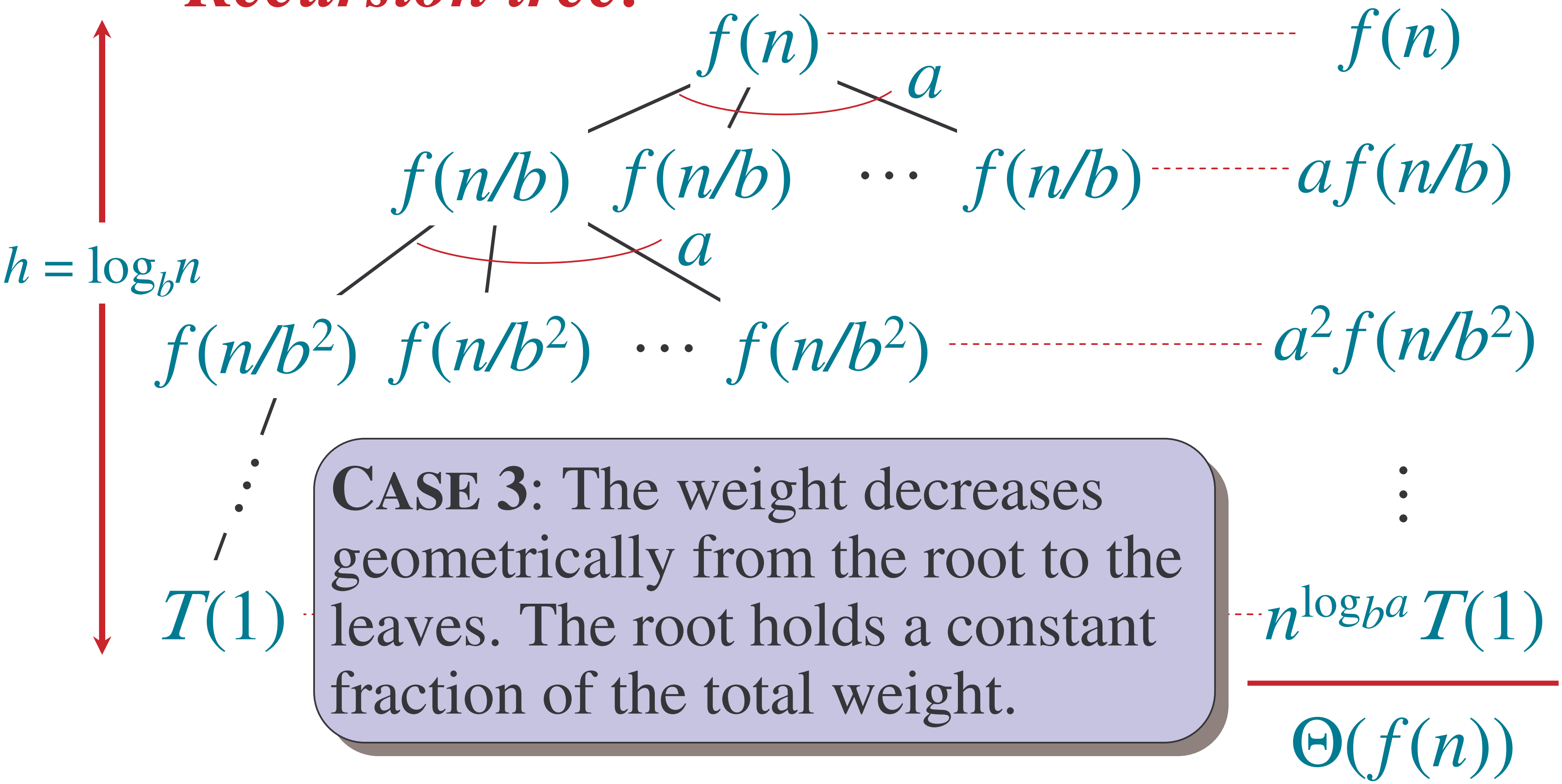
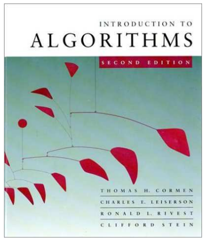
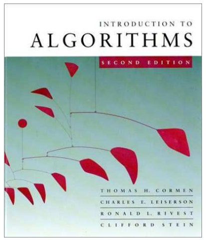


CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.



Examples

Ex. $T(n) = 4T(n/2) + n$
$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$
Case 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.
$\therefore\ T(n) = \Theta(n^2)$.



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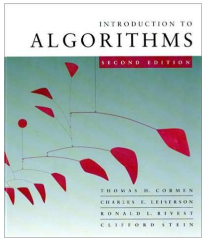
Ex. $T(n) = 4T(n/2) + n^2$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2.$

Case 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n^2 \lg n).$



Examples

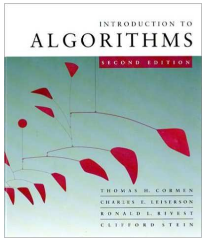
Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \leq cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3)$.



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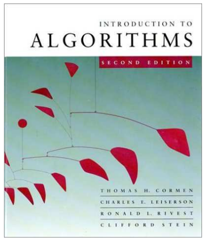
and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3)$.

Ex. $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2/\lg n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\lg n)$.



Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

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