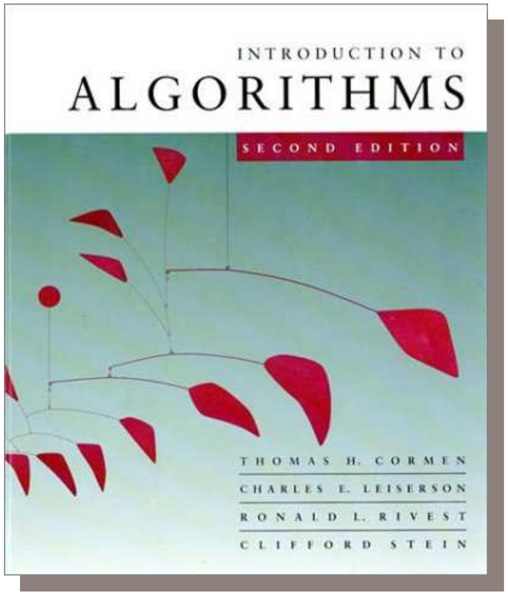
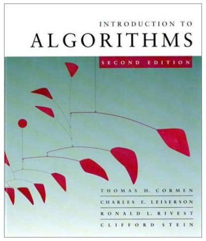
Data Structures and Algorithms
CSE 465



LECTURE 5
Solving Recurrences
- Substitution Method

Sofya Raskhodnikova and Adam Smith



Review questions

- Given a square matrix A and a n-bit integer b, how many matrix multiplications are needed to compute A^b in terms of n…

a. If we use the naïve algorithm using

$$A^b = A \times A \times \cdots \times A \quad (b \text{ times})?$$

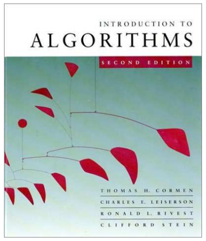
(Answer: $\Theta(2^n)$. The method requires $b-1$ multiplications. Since b is n bits long, it is between 2^{n-1} and 2^n-1.)

b. By the best algorithm you can come up with?

(Answer: $\Theta(n)$ is the best we know. Use exponentiation algorithm from last lecture, but multplying matrices instead of numbers.)

Methods for solving recurrences

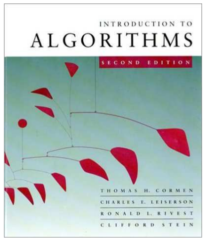
- Recursion tree
  - Examples in last two lectures
- Substitution method
  - This lecture
- Master theorem



Substitution method

The most general method:

1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.



Substitution method

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1. ***Guess*** the form of the solution.
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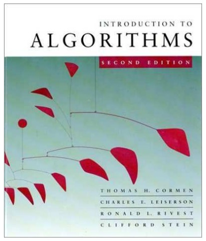
EXAMPLE: $T(n) = 4T(n/2) + n$

- [Assume that $T(1) = \Theta(1)$.]
- Guess $O(n^3)$. (Prove O and Ω separately.)
- Assume that $T(k) \leq ck^3$ for $k < n$.
- Prove $T(n) \leq cn^3$ by induction.

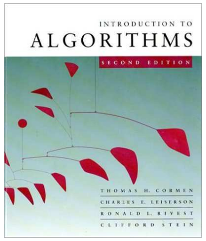


Example of substitution

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^3 + n \\
&= (c/2)n^3 + n \\
&= cn^3 - ((c/2)n^3 - n) \leftarrow \textit{desired} - \textit{residual} \\
&\le cn^3 \leftarrow \textit{desired}
\end{aligned}
$$

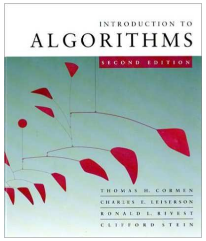
whenever $(c/2)n^3 - n \ge 0$, for example, if $c \ge 2$ and $n \ge 1$.

residual



Example (continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \le n < n_0$, we have "$\Theta(1)$" $\le cn^3$, if we pick c big enough.



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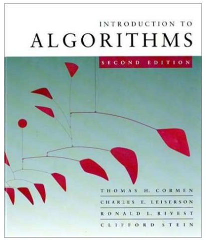
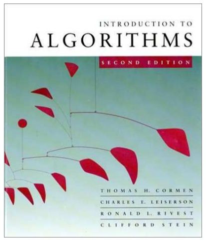
This bound is not tight!



A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

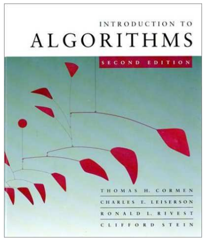


A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

Assume that $T(k) \leq ck^2$ for $k < n$:

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\leq 4c(n/2)^2 + n \\ &= cn^2 + n \\ &= O(n^2) \end{aligned}$$



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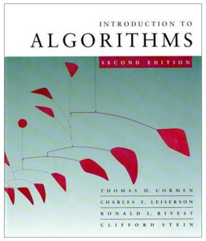
T(n) &= 4T(n/2) + n \\
&\leq 4c(n/2)^2 + n \\
&= cn^2 + n \\
&= O(n^2)
\end{aligned}
$$

Wrong! We must prove the inductive hypothesis



A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

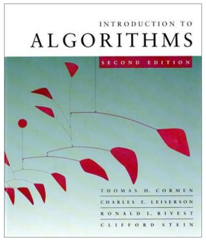
Assume that $T(k) \leq ck^2$ for $k < n$:

$$T(n) = 4T(n/2) + n$$
$$\leq 4c(n/2)^2 + n$$
$$= cn^2 + n$$

~~$= O(n^2)$~~ ***Wrong!*** We must prove the I.H.

$= cn^2 - (-n)$ [desired – residual]

$\leq cn^2$ for ***no*** choice of $c > 0$. Lose!

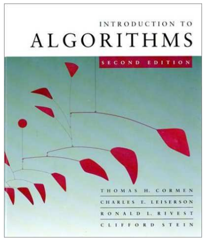


A tighter upper bound!

IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$.



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Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$.

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\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4(c_1(n/2)^2 - c_2(n/2)) + n \\
&= c_1 n^2 - 2c_2 n + n \\
&= c_1 n^2 - c_2 n - (c_2 n - n) \\
&\le c_1 n^2 - c_2 n \quad \text{if } c_2 \ge 1.
\end{aligned}
$$

A tighter upper bound!

IDEA: Strengthen the inductive hypothesis.

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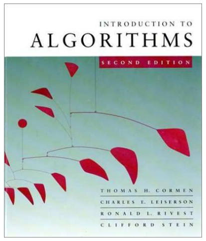
\end{aligned}
$$

Pick c_1 big enough to handle the initial conditions.



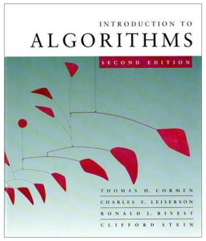
A matching lower bound

IDEA: Formulate inductive hypothesis with "$\geq$"

Inductive hypothesis: $T(k) \geq ck^2$ for all $k < n$.

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\geq 4\,cn^2/4 + n \\
&= cn^2 + n \\
&\geq cn^2 \text{ for any } c \geq 0.
\end{aligned}
$$

Pick c small enough to handle the initial conditions.



Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

Return to last slide viewed.