Lecture 10

Solving recurrences

• Master theorem

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Review questions

• Guess the solution to the recurrence:
  \[ T(n) = 2T(n/3) + n^{3/2}. \]
  (Answer: \( \Theta(n^{3/2}) \).

• Draw the recursion tree for this recurrence.
  a. What is its height?
    (Answer: \( h = \log_3 n \).)
  b. What is the number of leaves in the tree?
    (Answer: \( n^{(1/\log 3)} \).)
The master method applies to recurrences of the form

\[ T(n) = a T(n/b) + f(n) , \]

where \( a \geq 1 \), \( b > 1 \), and \( f \) is asymptotically positive, that is \( f(n) > 0 \) for all \( n > n_0 \).
Three common cases

Compare $f(n)$ with $n^{\log ba}$:

1. $f(n) = O(n^{\log ba - \varepsilon})$ for some constant $\varepsilon > 0$.
   • $f(n)$ grows polynomially slower than $n^{\log ba}$ (by an $n^\varepsilon$ factor).

   **Solution:** $T(n) = \Theta(n^{\log ba})$. 

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1. $f(n) = O(n^{\log ba} - \varepsilon)$ for some constant $\varepsilon > 0$.
   • $f(n)$ grows polynomially slower than $n^{\log ba}$ (by an $n^\varepsilon$ factor).

   \text{Solution:} \ T(n) = \Theta(n^{\log ba}) .

2. $f(n) = \Theta(n^{\log ba} \lg^k n)$ for some constant $k \geq 0$.
   • $f(n)$ and $n^{\log ba}$ grow at similar rates.

   \text{Solution:} \ T(n) = \Theta(n^{\log ba} \lg^{k+1} n) .
Compare $f(n)$ with $n^{\log_{b}a}$:

3. $f(n) = \Omega(n^{\log_{b}a} + \varepsilon)$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_{b}a}$ (by an $n^{\varepsilon}$ factor),

   and $f(n)$ satisfies the regularity condition that $af(n/b) \leq cf(n)$ for some constant $c < 1$.

Solution: $T(n) = \Theta(f(n))$.
Idea of master theorem

Recursion tree:

\[ f(n) \]
\[ f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b) \]
\[ f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2) \]
\[ \vdots \]
\[ T(1) \]
Idea of master theorem

Recursion tree:

\[ T(n) = \begin{cases} f(n) & \text{if } a = 1, \\ f(n) \leq \frac{af(n)}{b^2} & \text{if } a > 1, \\ f(n) \leq \frac{af(n)}{b} & \text{if } 0 < a \leq 1. \end{cases} \]
Idea of master theorem

Recursion tree:

\[ h = \log_b n \]

\[ T(1) \]

\[ f(n) \]

\[ f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b) \quad a f(n/b) \]

\[ f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2) \quad a^2 f(n/b^2) \]

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S. Raskhodnikova and A. Smith. Based on notes by E. Demaine and C. Leiserson
Idea of master theorem

Recursion tree:

\[ f(n) \]

\[ \frac{f(n)}{a} \]

\[ \frac{f(n/b)}{a} \]

\[ \frac{f(n/b^2)}{a} \]

\[ \vdots \]

\[ T(1) \]

\[ \text{#leaves} = a^h \]

\[ = a^{\log_b n} \]

\[ = n^{\log_b a} \]

2/6/12

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Idea of master theorem

**Recursion tree:**

\[ h = \log_b n \]

\[ f(n) \]

\[ f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b) \]

\[ \cdots \]

\[ a \]

\[ a^2 f(n/b^2) \]

\[ \vdots \]

\[ T(1) \]

\[ n^\log_b a \quad T(1) \]

\[ \Theta(n^\log_b a) \]

**CASE 1:** The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

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Idea of master theorem

Recursion tree:

\[ h = \log_b n \]

\[ f(n) \]

\[ a \]

\[ f(n/b) \]

\[ f(n/b) \]

\[ \cdots \]

\[ f(n/b) \]

\[ a f(n/b) \]

\[ a^2 f(n/b^2) \]

\[ \cdots \]

\[ n^{\log_b a} T(1) \]

\[ \Theta(n^{\log_b a} \log n) \]

CASE 2: \((k = 0)\) The weight is approximately the same on each of the \(\log_b n\) levels.
Idea of master theorem

Recursion tree:

\[ f(n) \]

\[ f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b) \quad a f(n/b) \]

\[ f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2) \quad a^2 f(n/b^2) \]

\[ T(1) \]

\[ n^{\log_b a} T(1) \]

\[ \Theta(f(n)) \]

**CASE 3:** The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

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Examples

Ex. \( T(n) = 4T(n/2) + n \)
\[ a = 4, \ b = 2 \Rightarrow n^{\log_b a} = n^2; \ f(n) = n. \]

Case 1: \( f(n) = O(n^2 - \varepsilon) \) for \( \varepsilon = 1. \)
∴ \( T(n) = \Theta(n^2). \)
Examples

Ex. \[ T(n) = 4T(n/2) + n \]
a = 4, b = 2 \( \Rightarrow \) \( n^{\log_b a} = n^2; f(n) = n. \)

Case 1: \( f(n) = \mathcal{O}(n^2 - \varepsilon) \) for \( \varepsilon = 1. \)

\[ \therefore T(n) = \Theta(n^2). \]

Ex. \[ T(n) = 4T(n/2) + n^2 \]
a = 4, b = 2 \( \Rightarrow \) \( n^{\log_b a} = n^2; f(n) = n^2. \)

Case 2: \( f(n) = \Theta(n^2 \lg^kn), \) that is, \( k = 0. \)

\[ \therefore T(n) = \Theta(n^2 \lg n). \]
**Examples**

**Ex.** \( T(n) = 4T(n/2) + n^3 \)

\( a = 4, \ b = 2 \Rightarrow n^{\log_b a} = n^2; \ f(n) = n^3. \)

**CASE 3:** \( f(n) = \Omega(n^2 + \varepsilon) \) for \( \varepsilon = 1 \)

and \( 4(n/2)^3 \leq cn^3 \) (reg. cond.) for \( c = 1/2. \)

\( \therefore T(n) = \Theta(n^3). \)
**Examples**

**Ex.** $T(n) = 4T(n/2) + n^3$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3$.

**Case 3:** $f(n) = \Omega(n^2 + \varepsilon)$ for $\varepsilon = 1$
and $4(n/2)^3 \leq cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore T(n) = \Theta(n^3)$.

**Ex.** $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2/\lg n$.

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^\varepsilon = \omega(\lg n)$.