Data Structures and Algorithms
CMPSC 465

LECTURE 4
• More Asymptotic Notation

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**o-notation and ω-notation**

*O*-notation and *Ω*-notation are like ≤ and ≥. *o*-notation and *ω*-notation are like < and >.

\[
o(g(n)) = \{ f(n) : \text{for any constant } c > 0, \text{ there is a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < c g(n) \text{ for all } n \geq n_0 \}\]

**Example:** \(2n^2 = o(n^3)\) \((n_0 = 2/c)\)
o-notation and ω-notation

$O$-notation and $Ω$-notation are like $\leq$ and $\geq$.

$o$-notation and $ω$-notation are like $<$ and $>$.  

$$ω(g(n)) = \{ f(n) : \text{for any constant } c > 0, \text{ there is a constant } n_0 > 0 \text{ such that } 0 \leq cg(n) < f(n) \text{ for all } n \geq n_0 \}$$

**Example:** $\sqrt{n} = ω(lg n)$  \( (n_0 = 1+1/c) \)
## Summary

<table>
<thead>
<tr>
<th>Notation</th>
<th>… means …</th>
<th>Think…</th>
<th>E.g.</th>
<th>Lim ( f(n)/g(n) )</th>
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</thead>
<tbody>
<tr>
<td>( f(n)=O(n) )</td>
<td>( \exists c&gt;0, n_0&gt;0, \forall n &gt; n_0 : 0 \leq f(n) &lt; cg(n) )</td>
<td>Upper bound “( \leq )&quot;</td>
<td>( 100n^2 = O(n^3) )</td>
<td>If it exists, it is ( &lt; \infty )</td>
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<tr>
<td>( f(n)=\Omega(g(n)) )</td>
<td>( \exists c&gt;0, n_0&gt;0, \forall n &gt; n_0 : 0 \leq cg(n) &lt; f(n) )</td>
<td>Lower bound “( \geq )&quot;</td>
<td>( n^{100} = \Omega(2^n) )</td>
<td>If it exists, it is ( &gt; 0 )</td>
</tr>
<tr>
<td>( f(n)=\Theta(g(n)) )</td>
<td>both of the above: ( f=\Omega(g) ) and ( f = O(g) )</td>
<td>Tight bound “( = )&quot;</td>
<td>( \log(n!) = \Theta(n \log n) )</td>
<td>If it exists, it is ( &gt; 0 ) and ( &lt; \infty )</td>
</tr>
<tr>
<td>( f(n)=o(g(n)) )</td>
<td>( \forall c&gt;0, n_0&gt;0, \forall n &gt; n_0 : 0 \leq f(n) &lt; cg(n) )</td>
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<td>( f(n)=\omega(g(n)) )</td>
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<td>“( &gt; )&quot;</td>
<td>( n^2 = \omega(\log n) )</td>
<td>Limit exists, ( =\infty )</td>
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</tbody>
</table>
Common Functions: Asymptotic Bounds

- **Polynomials.** $a_0 + a_1n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$.

- **Polynomial time.** Running time is $O(n^d)$ for some constant $d$ independent of the input size $n$.

- **Logarithms.** $\log_a n = \Theta(\log_b n)$ for all constants $a, b > 0$.

  - Can avoid specifying the base.

  - Log grows slower than every polynomial.

  For every $x > 0$, $\log n = O(n^x)$.

- **Exponentials.** For all $r > 1$ and all $d > 0$, $n^d = O(r^n)$.

- **Factorial.** $n! = (\sqrt{2\pi n}) \left(\frac{n}{e}\right)^n (1 + o(1)) = 2^{\Theta(n \log n)}$

  - Every exponential grows faster than every polynomial.

  - Grows faster than every exponential.

A. Smith; based on slides by E. Demaine, C. Leiserson, S. Raskhodnikova, K. Wayne
Properties

• Transitivity:
  – If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

• Additivity:
  – If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.

• Multiplication:
  – If $f = O(h_1)$ and $g = O(h_2)$ then $f(n)g(n) = O(h_1(n)h_2(n))$

• Similar properties for $\Theta$, $\Omega$, $o$, $\omega$
Exercise: Show that \( \log(n!) = \Theta(n \log n) \)

- **Upper bound:**
  \[
  \log(n!) = \sum_{i=1}^{n} \log(i) \\
  \leq n \log(n) = O(n \log n)
  \]

- **Lower bound:**
  \[
  \log(n!) = \sum_{i=1}^{n} \log(i) \\
  \geq \sum_{i=\lceil n/2 \rceil}^{n} \log(i) \\
  \geq \frac{n}{2} \log\left(\frac{n}{2}\right) \geq \frac{n}{2} \log\left(\frac{n}{2} - 1\right) \\
  = \frac{n}{2} \log\left(\frac{n}{2}(1 - \frac{n}{2})\right) \\
  = \frac{n}{2} \log(n)\left(1 - \frac{\log 2}{\log n} + \frac{\log(1 - \frac{n}{2})}{\log n}\right) \\
  = \Omega(n \log n) \cdot \Omega(1) = \Omega(n \log n)
  \]
Review questions: True/false?

1) $n^2 - 5n - 100 = O(n)$  
2) $n^3 + 10n^2 + 125 = \omega(n)$  
3) $n^2 + O(n) = O(n^2)$  
4) $2^{n+1} = O(2^n)$  
5) $2^{5n} = O(2^n)$  
6) $\log(n^2) = O(\log(n))$  
7) $5n - O(n) = \Omega(n)$  
8) $5n - o(n) = \Omega(n)$

1) F  2) T  3) T  4) T  5) F  6) T  7) F  8) T