I. Independent Poisson Collection and Coupons-Reset Processes

We consider a coupon collector that selects from a set of \( M \) different coupons uniformly at random and with replacement. Coupon selection is a stationary Poisson process with rate \( \beta \). All \( M \) coupons are periodically changed/reset according to an independent Poisson process with rate \( \delta \). When the coupons change, the collector starts over with zero currently valid coupons. Let

\[ z = \frac{\beta}{\delta}. \]

**Proposition 1:** The stationary (viewed at a typical time) mean number of different coupons obtained is

\[ \frac{Mz}{M + z}. \]  

**Proof:** Suppose that, starting from zero, the collector selects coupons over a period of time \( T \) during which the coupons do not change. If \( K \) coupons are selected during this time, then the mean number of different coupons obtained is

\[ M(1 - (1 - 1/M)^K). \]

Since \( K \sim \text{Poisson}(\beta T) \), we can condition on \( K \) to get that the mean number of different coupons obtained is simply

\[ \sum_{K=0}^{\infty} M(1 - (1 - 1/M)^K) \frac{(\beta T)^K e^{-\beta T}}{K!} = M(1 - e^{-\beta T/M}). \]  

If \( T \sim \text{exp}(\delta) \) then the expected number of coupons found just before they change (i.e., the “average peak”) is, by simply conditioning on \( T \),

\[ \int_0^\infty M(1 - e^{-\beta T/M}) e^{-\delta T} dT = \frac{Mz}{M + z}. \]

Finally, since Poisson Arrivals See Time Averages (PASTA) [2], [1], the stationary view is the same as that just before the coupons change.
II. CONSTANT-RATE COLLECTION AND POISSON COUPONS-RESET PROCESSES

Under constant rate coupon-selection (constant rate $\beta$) and Poisson coupon-changing process (rate $\delta$), the stationary mean number of different currently valid coupons obtained by the botnet is approximately

$$\frac{Mz}{v^{-1} + z}$$

(3)

where $v = -\log(1 - 1/M) > 0$.

To see why, note that at constant $\beta$ selections/s, the number of coupons selected over $[0,T]$ is simply $L = \lfloor \beta T \rfloor \approx \beta T$. So, using the moment generating function of $T$ we get,

$$E(M(1 - (1 - 1/M)^{L})) \approx EM(1 - (1 - 1/M)^{\beta T})$$

$$= EM(1 - e^{-Tv\beta})$$

$$= M \left( 1 - \frac{\delta}{\delta + \beta v} \right).$$

The exact expression for constant coupon selection is

$$\frac{1}{e^{1/z} - (1 - 1/M)}.$$  

(4)

Note that (3) and (4) are close for large $z$ and $M$ because $v \sim 1/M$ and $e^{1/z} \sim 1 + 1/z$. Also note that (1) and (4) are close for large $z$.

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REFERENCES
