Multicommodity aggregative games for tenant orchestration in a public, neutral cloud

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I. INTRODUCTION

In this report, we give an overview of multicommodity aggregative models of resource management by both tenants and public, neutral clouds. A tenant will use an estimate of a near-future net valuation to make decisions regarding IT resource allocation. We study when equilibrium (when tenant demands do not exceed supply) and of statistical multiplexing are given. We study the interior Nash equilibrium (when tenant demands do not exceed supply) and how “fair” distribution of resources may be achieved when demand exceeds supply, i.e., under congestion.

II. NOTATION AND BRIEF OVERVIEW OF NONCOOPERATIVE GAMES

Consider a non-cooperative, N-consumer game where consumer $i$’s net utility/valuation is the function $u_i(x_i; x_{-i})$, $x_i$ is consumer $i$’s play-action (incident workload), and $x_{-i}$ is the N-vector $x$ without $x_i$. The performance of a system under a noncooperative game is often assessed at Nash equilibrium. A Nash equilibrium is a play-action $x^*$ such that for all tenants $i$, $x_i^* = \arg \max_{x_i} u_i(x_i; x_{-i}^*)$, i.e., any unilateral deviation from Nash equilibrium cannot benefit the deviating player. Define the operator $\partial_i u_i := \frac{\partial u_i}{\partial x_i}$. In the case where

$$u_i(x_i; x_{-i}) \text{ is concave in } x_i \text{ for all } i, x_{-i}; \quad (1)$$

the first-order necessary condition (FONC) $(\partial_i u_i)(x_i^*(x_{-i}); x_{-i}) = 0$, if feasible sufficiently defines the best response for all $i$. In this paper, we simplify matters by assuming continuous optimization but results are easily adapted to deal with discrete play-action spaces.

For an iterative game, the (best-response) play-action of tenant $i$ in iteration $k + 1 \in \mathbb{Z}^+$ is

$$x_i^{(k+1)*} := \arg \max_{x_i} u_i(x_i; x_{-i}^{(k)*}). \quad (2)$$

Under continuity assumptions of the best-response mapping, a (possibly boundary) Nash equilibrium exists by Brouwer’s fixed-point theorem [3], or if the best-response is multivalued, then Kakutani’s fixed point theorem can be used. Results on uniqueness of Nash equilibrium and convergence to it by an iterative game are known under concavity assumptions on the net valuation and sufficiently small step-size, i.e., “better response” [16], [12] or “continuous best reply”. For the latter, see [29] Theorems 7-9 and [29] Theorem 3 (as summarized in [23]).

If a potential (incremental improvement) function [19] is available, then existence and convergence to a Nash equilibrium can be shown for a finite, discrete game [25], [11], or for the synchronous better-response game with sufficiently small step sizes (the latter by arguing as for stability in the sense of Lyapunov), cf., Appendix B. Sometimes a Lyapunov function is known when an exact potential is not, e.g., for the (non-aggregative, plural equilibria) ALOHA game, or for approximate best-response (yielding the same equilibria) for an aggregative Erlang-loss game [12]. Simple examples with plurality of Nash equilibria having different stability and Pareto optimality properties are given in [12]. A simple example wherein a Nash equilibrium does not exist is given in Figure 1 of [9].

III. TENANTS WITH LIMITED RESOURCE DEMANDS

A. A single-resource aggregative game

Consider the case of an aggregate game [10], [11] with all utilities of the form

$$\forall i, x_i, u_i(x_i) = u_i(x_i; s_{-i}), \text{ where } s_{-i} := \sum_{j \neq i} x_j. \quad (3)$$

That is, for aggregate games we replace $x_{-i}$ with $s_{-i}$. For a non-cooperative game involving a single shared resource, the resource consumed could be a “bottleneck” IT resource, or it could be a lumped/amalgamated resource such as an energy measure (e.g., [20]) or a number of equally provisioned VMs. For example [20], suppose that the play-actions $x$ directly relate to a single shared resource of amount $S$ and that $\forall i$, tenant $i$’s net valuation

$$u_i(x_i; s_{-i}) = v_i(x_i) - \frac{x_i}{x_i + s_{-i}} p(s_{-i} + x_i) =: v_i(x_i) - x_i \hat{p}(s_{-i} + x_i) \quad (4)$$

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1 e.g., CPU, RAM, I/O including egress network bandwidth, VLAN service, or portion/slice thereof
2 The FONC for interior Nash equilibria are generalized to variational inequalities to accommodate boundary equilibria [5].
where valuation function \( v_i \) is increasing and concave while the cost function \( p(s) := s\tilde{p}(s) \) is increasing, convex and shared proportionately to consumption.

**B. A multicommodity aggregative game with additive resource costs**

We now consider the more general case of a group of \( K \geq 1 \) different resources, where \( R_k \) is the amount of available resource of type \( k \). Assume as in [8], [21] that for each tenant \( i \), resources are consumed in fixed proportions according to the \( K \)-vector \( d_i \) so that each tenant \( i \) continues to have just a single play-action \( x_i \), typically interpreted as job-rate, generating consumption \( x_i d_{i,k} \) of resource \( k \). Here, \( \forall i, k \)

\[
  s_{-i,k} := \sum_{j \neq i} d_{j,k} x_j \quad \text{and} \quad s_k := \sum_j d_{j,k} x_j \leq R_k. \tag{5}
\]

Consider the vector \( \underline{s}_{-i} \) with components \( s_{i,k} \) corresponding to each resource.

Assuming additive costs across different resources, we define the net utilizations/valuations as

\[
  u_i(x_i; \underline{s}_{-i}) := v_i(x_i) - x_i \sum_k d_{i,k} \tilde{p}_k(s_{-i,k} + x_i d_{i,k})(6)
\]

See Appendix A for an example with quadratic models for prices and valuations.

**C. Load balancing**

A load-balancing problem for near-future workload can be framed

- for a cloud among its tenants in a single datacenter or among its (geographically disperse) datacenters, or
- for a tenant among its different VMs (and different “containers” within them handling different workload types), or among different cloud service providers.

Suppose at different sites \( w \in \mathcal{W} \), IT resources are collocated (pooleled). Again, we assume resource needs \( d_{i,k} \) are known or can be short-term predicted with accuracy. The incident workload of consumer/tenant \( i \) is now a \( |\mathcal{W}| \)-vector, \( \underline{x}_i = \{x_{i,w}\} \), so that its consumed resources of type \( k \) and site \( w \) are \( x_{i,w} d_{i,k} \), and its total incident workload is \( x_i := \sum_w x_{i,w} \). So, we can write the net valuation of tenant \( i \) as

\[
  u_i(\underline{x}_i) = v_i(\sum_w x_{i,w} q_{i,w}) - \sum_w x_{i,w} \sum_k d_{i,k} \tilde{p}_k(\sum_j d_{j,k} x_{j,w}),
\]

where here the positive parameter \( q_{i,w} \leq 1 \) is a relative QoS factor associated with site \( w \) (there are other ways to model the impact of QoS degradation on valuation).

We can moreover consider a plurality of independent workflows \( j \) for each tenant \( i \), i.e., \( x_{i,j,w} \) and associated resource-requirement parameters \( d_{i,j,k} \) leading to total net valuations \( \sum_j u_{i,j} \) where

\[
  u_{i,j} = v_{i,j}(\sum_w x_{i,j,w} q_{i,j,w}) - \sum_w x_{i,j,w} \sum_k d_{i,j,k} \tilde{p}_k, \tag{7}
\]

where the different sites in question could be different cloud providers, in which case the prices may also be site dependent, i.e., \( p_{k,w} \). A commonly addressed such problem involves choosing between proximal computational and storage resources that are congested, or distant resources that are not but have higher associated networking costs/delays (modeled above through the \( q \) parameters).

Note how load balancing includes the possibility of removing containers (setting \( x_{i,j,w} = 0 \) at some site \( w \)) or adding containers (increasing \( x_{i,j,w} \) to positive from zero). Indeed, if all containers in a VM are assigned no workload then the VM is essentially removed, and a VM can be similarly initiated when all of its containers are simultaneously added.

For some types of work, different resource vectors \( \underline{d} \) can be used to achieve the same QoS for the same workload, \( x \). Modeling this flexibility as a domain of such acceptable demand resources, \( D \), it may be possible that tenants can also maximize their net utilities \( \underline{x} \) over \( \underline{d} \in D \).

**D. Exploiting statistical multiplexing by the tenant**

Statistical multiplexing may manifest when increasing the intensity of a single type of workload. If \( x \) is the incident workload intensity, then we can define the function \( D(x) \) as the resources needed to equate the required QoS \( q \) to the received QoS: \( Q(x, D(x)) = q \). Overbooking (or statistical multiplexing) is possible for a given workload type when \( D \) is strictly sub-additive: \( D(\xi_1 + \xi_2) < D(\xi_1) + D(\xi_2) \) whenever \( \xi_1, \xi_2 > 0 \); in particular, not linearly proportionate as \( D(x) = dx \) for some constant \( d \) as assumed above.

For a simple example, consider the mean delay of an M/M/1 queue with mean job service rate \( D \) and workload (mean job arrival rate) \( x < D \), so that the mean sojourn time (queueing delay plus service time) is \( Q = 1/(D - x) \). Fixing the QoS \( Q = q \), we see here that \( D(x) = x + q^{-1} \) is affine and thus sub-additive.

So, in \( \underline{5} \) and \( \underline{6} \) replace \( x_i d_{i,k} \) with the concave, non-decreasing function \( D_{i,k}(x_i) \) satisfying \( D_{i,k}(0) = 0 \):

\[
  u_i(\underline{x}) = v_i(x_i) - \sum_k D_{i,k}(x_i) \tilde{p}_k \left( \sum_j D_{j,k}(x_j) \right).
\]

Note that the cloud may exploit statistical multiplexing between tenants, i.e., the resource of type \( k \) required to accommodate the tenants at demand \( \underline{x} \) may be less than \( \sum_j D_{j,k}(x_j) \), cf., Section [IV-C]

**E. Discussion: Interacting containers and VMs: network resources and workload heterogeneity within application**

One can model how a tenant may load-balance resources among different containers within a single VM using a criterion like \( \underline{7} \) but at one site \( w \in \mathcal{W} \) and assuming independent workload types \( j \). But it’s often the case that multiple containers in a VM are interacting (communicating) to execute tasks that are part of the same sequence of jobs, e.g., two mapper containers or a mapper and reducer container that are part of the same MapReduce application.
Communicating VMs can be on the same rack or different racks of the same physical machine, or on different physical machines. VM or container I/O, particularly egress network bandwidth reservations, could be allocated \([17], [6], [13], [7]\) and more explicitly accounted as one of the IT resources \(k\) via the cost functions \(\tilde{p}_k\) (as in constraint based routing). Networking resources could also be allocated to Classes-of-Service, as in the diffserv framework, that may involve class-differential resource-access priorities.

A complication is workload heterogeneity within a single application spanning a group of interacting containers (as in MapReduce \([28], [4]\)), unlike \([7]\) where each workload type \(j\) independently uses different IT resource pools. That is, each aggregate workload will be \textit{composed} of a weighted sum of different workloads each with its own characteristic resource \((d)\) requirements.

Also, in an application spanning multiple containers, not all containers will have \textit{exogenous} workload demand \(x_{i,j,w}\). Given acceptable service quality (e.g., within total response-time bounds) for jobs, we expect valuations for each job type \(j\) to remain a function of total throughput, \(\sum_w x_{i,j,w}\). But here there are potentially more sites \(w\) to consider whose performance will impact this total throughput. These “internal” sites (e.g., reducers in the MapReduce framework) will naturally add additional terms to the cost side of the tenant net valuation/utility expressions above. The workload factors for those internal containers can be computed by, e.g., use of flow-balance equations as for stable networks of queues.

Given persistent QoS problems, the tenant may diagnose resource inadequacies/bottlenecks at some site and, equipped with online estimates of sensitivities of overall throughput to changing resource allocations at different sites, act to rebalance resources or modify (local) demand accordingly. Again, this can guided by benefit-cost criteria such as the tenant net valuations, \(u\).

In particular, the tenant can consider migrating containers within its base of VMs to deal with variable network congestion/costs \([2]\). For example, if delays through the network (shuffler) between a particular mapper and not collocated reducer are diagnosed as problematic, the tenant can consider consolidating (collocating) these two containers in one VM.

IV. RESOURCE MANAGEMENT BY THE PUBLIC CLOUD

A. When tenant demands exceed available resources

Tenant demand will exceed resources when prices are low\(^4\). In this section, we will simply consider linear valuations,

\[
\forall i, \quad v_i(x_i) = V_i x_i \quad \text{for constants } V_i,
\]

and linear cost functions resulting in constant \(\tilde{p}_k(s_i - k + x_i d_{i,k}) \equiv \tilde{p}_k\) for all resources \(k\), leading to \textit{assumed positive} linear net-utilities as follows:

\[
u_i(x_i) = V_i x_i - x_i \sum_k d_{i,k} \tilde{p}_k = x_i (V_i - \sum_k d_{i,k} \tilde{p}_k)
\]

Note here the valuation of tenants denoted as the scalar \(V_i\) indicates tenant \(i\)’s willingness to pay or priority, and resource costs \(\tilde{p}_k\) indicate their priority allowing them to be comparatively valued, e.g., \(\tilde{p}_k = 1/R_k\) for kind of asset fairness \([8]\).

Figure 1 depicts illustrative example (A) with

- total resource pool \(R = (9\text{ CPUs}, 6\text{GB RAM})\) and
- tenant demand vectors \(d_1 = (1, 2), d_2 = (3, 1)\).

A game set-up with linear utilities and valuations such that

\[\forall i, \quad \partial_i u_i = V_i - \sum_k d_{i,k} \tilde{p}_k > 0\]

will result in a set of Nash equilibria on the feasibility boundary where at least one resource is exhausted by tenants\(^4\).

A leader of the game (operator/provider of the cloud itself, or a government/market regulator) may desire equality points that, e.g., maximize: cloud revenue, \(\Omega := \sum_i x_i \sum_k d_{i,k} \tilde{p}_k\); social welfare, \(\Omega := \sum_i u_i(x_i)\); or total tenants’ valuations, \(\Omega := \sum_i V_i x_i\).

To such ends, resources may be shifted between tenants to control their Nash equilibrium, \textit{i.e.}, the cloud takes direct resource allocation actions at or near the feasibility region (as in the CEEI mechanism \([8]\)). Note that the three previous example objectives are planar. They result in (collective) play-actions \((x)\) with maximal \(\Omega\) corresponding to corner points of the feasibility region, recall the simplex algorithm \([24]\).

Alternatively, resources \(\bar{x}\) can be (maximally) allocated subject to “fairness” constraints. For example, equal dominant resource share, \(x_i \max_r d_{i,r}/R_r\) (leading to dominant resource fairness, DRF) \([8]\); equal total asset-fraction share, \(x_i \sum_r d_{i,r}/R_r\) (leading to a kind of asset fairness); or equal per unit-net-valueation, \(x_i/(V_i - \sum_r d_{i,r} \tilde{p}_r)\).

\(^4\)Recall that any unilateral defection from a Nash equilibrium cannot result in increased utility for the defector. At the convex feasibility boundary with at least one resource exhausted, the only feasible unilateral move by any player \(i\) is to reduce demand \((x_i)\), hence utility \(x_i \partial_i u_i\) is reduced if, as assumed, marginal utility \(\partial u_i > 0\) for all \(i\).
Note that notions of fairness may not separately consider the interests of the cloud and tenants particularly in a public, for-profit cloud setting with potentially competing tenants (possibly serving their own customers). Also, tenants can be differently weighted (as considered in e.g., Sec. 4.3 of [8]); such weights could correspond to priority in a private cloud or enterprise network, or willingness-to-pay/valuations in a public cloud system.

See Figure 2 for illustrative example (A) indicating DRF and maximum tenant valuation. For DRF, the dominant resource shares \( x_1 \max_k \frac{d_{i,k}}{R_k} \) of the two tenants \( i \in \{1,2\} \) are equated - in this case, \( x_1 2/6 = x_2 3/9 \), i.e., \( x_1 = x_2 \).

**B. Mechanism design for a public, neutral cloud**

One can formulate “mechanism design” problems to find resource allocation or pricing/auction (incentive) frameworks that lead to (are compatible with) certain desirable properties of Nash equilibrium, such as some measure of fairness (e.g., DRF, asset fairness) or some efficiency measure based on social welfare (e.g., [14]). For example, the cloud could adapt the Additive Increase, Multiplicative Decrease (AIDM) approach of TCP congestion control to this setting, here via dynamic SLA renegotiations or via some dynamic resource allocation protocols specified in the SLAs. When collective tenant demand exceeds feasible resources, the cloud signals the tenants that their resource allocations \( x_i d_{i,r} \) will be reduced by a (positive) factor \( \varepsilon < 1 \), i.e., \( \forall i,r, x_i d_{i,r} \rightarrow \varepsilon x_i d_{i,r} \). The tenants will correspondingly reduce their workloads \( x \).

Hereafter, the tenants collectively increase their resource demands, but only by incremental amounts as approved by the public cloud so that \( x_i d_{i,r} \rightarrow x_i d_{i,r} + \delta \eta_{i,r} \), where \( \eta_{i,r} > 0 \) and positive \( \delta \ll 1 \). Using the argument of R. Jain for TCP congestion control (e.g., [15]), in the limit the boundary Nash equilibrium \( x^* \) is such that \( x^*_i d_{i,r} \propto \eta_{i,r} \). For example, set \( \eta_{i,r} = d_{i,r} (V_i - \sum_k d_{i,k} P_k) / \max_k d_{i,k} \) for resource allocations proportional to net valuations under additive pricing, or set \( \eta_{i,r} = d_{i,r} R_k / \max_k d_{i,k} \) for DRF (with \( P_k = 1/R_k \)).

**C. Chance constraints and statistical multiplexing**

Now let \( m_i \) and \( \sigma_i^2 \) respectively be mean and variance of tenant \( i \)'s demand vector. Also, let \( \sigma_{i,j,k}^2 \) is the covariance of (demand of tenant \( i \) and tenant \( j \) for resource \( k \)) and define the covariance matrix \( C_k = [\sigma_{i,j,k}^2] \). We can extend our model of resource allocation by using nonlinear “chance constraints” for each resource \( k \), leading to a convex feasibility region:

\[
\mathbf{z}^T d_k + n_k \sqrt{\mathbf{z}^T C_k \mathbf{z}} \leq H_k < R_k, \tag{9}
\]

where \( n_k \geq 1 \) is a confidence factor for the headroom \( R_k - H_k \) corresponding to \( \text{Pr}(\sum_i x_i m_{i,k} \geq H_k) \leq \varepsilon_k \), and \( \forall k, x_k \geq 0 \) of course. Run-time estimates of mean, variance and covariance of demand and dynamic calibration of resource headroom can be jointly used to deal with uncertain, time-varying demand needs, particularly when infeasible overages in demand must be rare (\( \varepsilon_k \ll 1 \)). Headroom will be important in the presence of estimation error in these statistical parameters of demand.

If we here define

\[
d_{i,k} := m_{i,k} + n_k \sigma_{i,k}, \tag{10}
\]

(a quantities that could be involved in tenant \( i \)'s service level agreement with the cloud), we can then define statistical multiplexing gain for resource \( k \) at demand \( \mathbf{z} \) as

\[
n_k \left( \sum_i x_i \sigma_{i,k} - \sqrt{\mathbf{z}^T C_k \mathbf{z}} \right),
\]

i.e., capturing the difference between “deterministic” provisioning by the cloud using the \( d_{i,k} \) and that using \( \mathbf{z} \). We can define aggregate demand \( \mathbf{z} \) for resource \( k \) as “negatively correlated” when

\[
\mathbf{z}^T C_k \mathbf{z} < \sum_i x^2_i \sigma^2_i.
\]

In this case, or the case where the workload demands are uncorrelated (i.e., \( C_k \) is diagonal), we see that the statistical multiplexing gain is strictly positive by sub-additivity of square root. Even in cases where demand has small but strictly positive correlation, there is a possibility for statistical multiplexing gain.

Again note that statistical multiplexing gains can also be exploited by the tenant (containers within a VM or different VMs collocated on a physical machine).

In practice, it can be reckoned at run-time (online) whether different workloads are negatively correlated (i.e., \( C_k \) is diagonal), we see that the statistical multiplexing gains can be achieved by scaling a given workload [13, 27]. A basic assumption here is that workloads are sufficiently stationary so that present estimates of correlation are valid in the near future. Moreover, correlations need to be assessed jointly among the plurality IT resources used by the workloads in question.

Given such correlations, obviously the cloud would need to consider how best to situate tenant VMs among its existing

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5Recall that covariance matrices are always positive semi-definite with positive diagonal entries. Using the fact that covariance matrices are also symmetric, convexity of the feasibility region is a direct consequence of the Cauchy-Schwarz inequality, and that the intersection of convex regions (one for each resource \( k \)) is convex.
servers (workload consolidation), considering not only their internal IT resources but how they are rack-assigned and networked. Without the necessary characterizations of tenant workloads and their correlations, a cloud may heuristically engage in “trial underbooking” of resources and monitor the resulting tenant demand response. One aim of this practice would be to minimize the number of active servers and thereby lower operational costs, particularly energy related. However, as demand varies over time, servers may need to be turned on, which takes time and power.

REFERENCES


APPENDIX A: EXAMPLE OF QUADRATIC UTILITIES WITH UNIQUE INTERIOR NASH EQUILIBRIUM

Consider a quadratic pricing model for all resources:

\[ p_k(s_k) = \bar{p}_\text{min,k} s_k + \bar{p}_\text{max,k} s_k^2 / R_k \]

where all parameters are positive, and \( \bar{p}_k := \bar{p}_\text{max,k} / R_k \). Note that the pricing component of the utilities (10) are concave. Also assume the valuations are quadratic,

\[ v_i(x_i) = -a_i x_i^2 + b_i x_i \]

with \( 0 \leq x_i \leq b_i / (2a_i) \). (12) noting that \( v_i(0) = 0 \) and is concave. So, at interior Nash equilibrium: \( \forall i, = 0 = \partial_i u_i(x_i; \xi^*_{-i}) \), i.e.,

\[ 0 = \beta_i - 2\alpha_i + \sum_k \pi_k d_{i,k} x_i^* - \sum_j \left( \sum_k d_{i,j,k} \pi_k \right) x_j^* \]

where \( \beta_i := b_i - \sum_k d_{i,k} \bar{p}_\text{min,k} \). Let \( M_{i,i} := 2\alpha_i + \sum_k \pi_k d_{i,k}^2 \) and, \( \forall i \neq j, M_{i,j} := \sum_k d_{i,j,k} \pi_k = M_{j,i} \), and define the vector \( \beta := [\beta_i] \) (assumed \( \geq 0 \)) and the matrix \( M = [M_{i,j}] \). So, the workloads at Nash equilibrium \( \xi^* \) satisfy:

\[ M \xi^* = \beta \]

The following claim gives a condition for which (13) has an interior Nash equilibrium.

Claim 1. For sufficiently large \( R_k \), sufficiently small \( a_i \), and \( \beta \) is sufficiently close to a Perron eigenvector of \( M \), there exist unique interior Nash equilibria satisfying (13).

Proof: Clearly, for the Nash equilibrium \( \xi^* \) to be interior requires

\[ \forall k, \sum_i x_i^* d_{i,k} < R_k \]

and \( \forall i, 0 < x_i^* \leq b_i / (2a_i) \). If a finite solution to (13) exists and the \( R_k \) are sufficiently large, we can uniquely meet this first condition. So, we need to show a solution exists and the second condition.

By the Perron-Frobenius theorem for matrices with positive entries (like \( M \)) and by the continuity of solutions of linear equations: if \( \beta \) close to a Perron eigenvector of \( M \), then the solution to (13) is real and strictly positive and close to \( \beta / \lambda \), i.e., \( \xi^* \approx \beta / \lambda \), where \( \lambda \) is the (unique, positive, real) Perron eigenvalue of \( M \).

Additionally, if \( \max_i 2a_i < \lambda \) and \( \beta \) sufficiently close to a Perron eigenvector, then \( \forall i, x_i^* \leq \beta_i / (2a_i) < b_i / (2a_i) \).

\(^{a}\)All resource constraints have slackness.
Remarks on Claim 1. That \( \beta \) is close to a Perron eigenvector can be taken as a condition on the \( b_i \) parameters, noting that there are infinitely many Perron eigenvectors, all colinear and so they can be arbitrarily scaled by a positive amount. The condition \( \max_i 2a_i < \lambda \) can be met by suitable choice of the other parameters, \( d_{i,k} \) and \( \pi_k \) of \( M \). Obviously, if \( M \) is also nonsingular, then the Perron eigenvector \( \beta / \lambda \) is the unique Nash equilibrium.

APPENDIX B: POTENTIAL OF THE MULTICOMMODITY AGGREGATIVE GAME

We now consider the case where the synchronous game has an exact potential function (again, which can be used to show existence of and convergence to a Nash equilibrium [19], [25], [11]). First define “better response” iterative, synchronous existence of and convergence to a Nash equilibrium [19], [25], \( \forall \)

\( x \)

that cannot be canceled by the other two terms on the right-hand-side), and this for an arbitrary \( j \neq i \); hence it must be constant (mapping \( \mathbb{R}^K \to \mathbb{R} \)), \( m_i := \nabla x_i^* (s_{-i}) \). Moreover, \( \partial_j B_i(x_{-i}) + x_j \) must be constant in \( x_j \). Thus, for all \( x, i \neq j \):

\[
\partial_j \Phi(x) = -x_j + x_i d_j(m_i) + b_i, j(x_{-i}, -j),
\]

for some \( b_{i,j}(x_{-i}, -j) \) that does not depend on \( x_i \) or \( x_j \). Since \( \partial_j \partial_i \Phi = \partial_i \partial_j \Phi, d_j(m_i) = d_j(m_j) =: \kappa \neq i, j \) for all \( i \neq j \).

Thus,

\[
\Phi(x) = -\frac{x_j^2}{2} - \frac{x_i^2}{2} + \kappa_i x_i x_j + \beta_j x_j + \beta_i x_i + f_{i,j}(x_{-i}, -j),
\]

for some \( f_{i,j}(x_{-i}, -j) \) and constants \( \beta \). So, we arrive at (15) by induction.

Recall tenant criterion (6) with valuation function \( v_i \) increasing and non-convex and cost functions \( p_i(s) := s \beta_i(s) \) increasing and non-concave.

**Corollary 1.** If costs are quadratic [11], valuations are linear [8], and

\[
\sum_k d_{i,k}^2 \pi_k
\]

is a constant function of \( i \),

(16)

then the game based on the utilities (6) has an exact potential.

This corollary can be directly verified by solving \( \partial_i u_i = 0 \) for \( x_i \) and comparing the result to \( \partial_i \Phi + x_i \) using (15). Condition (16) is needed to achieve \( \kappa_{i,j} = \kappa_{j,i} \) for all \( j \neq i \).

\[\text{that cannot be canceled by the other two terms on the right-hand-side), and this for an arbitrary } j \neq i; \text{ hence it must be constant (mapping } \mathbb{R}^K \to \mathbb{R}, \text{ it is the unique Nash equilibrium.} \]

\( x \)

the consistent gradients condition, here

\[
\forall i, \quad \partial_i \Phi = x_i^* (s_{-i}) - x_i
\]

implies that for all \( i \), \( \partial_i \Phi(x) + x_i \) is not a function of \( x_i \), only a function of \( s_{-i} \). So, there are functions \( B_i \) depending only on \( s_{-i} \) such that

\[
\forall i, x, \quad \Phi(x) = -x_i^2 / 2 + x_i^* (s_{-i}) x_i + B_i(x_{-i}).
\]

Thus, for all \( j \neq i \),

\[
\partial_j \Phi(x) + x_j = d_j(m_i) x_i + \partial_j B_i(x_{-i}) + x_j
\]

also does not depend on \( x_j \). So, \( x_i^* (s_{-i}) \) cannot depend on \( x_j \) (because otherwise the factor \( x_i \) will then create a term

Another differential game is continuous best reply, \( \forall i, \partial_l u_i \) [22], [30], [23].

\[\text{Remarks on Claim 1. That } \beta \text{ is close to a Perron eigenvector can be taken as a condition on the } b_i \text{ parameters, noting that there are infinitely many Perron eigenvectors, all colinear and so they can be arbitrarily scaled by a positive amount. The condition } \max_i 2a_i < \lambda \text{ can be met by suitable choice of the other parameters, } d_{i,k} \text{ and } \pi_k \text{ of } M. \text{ Obviously, if } M \text{ is also nonsingular, then the Perron eigenvector } \beta / \lambda \text{ is the unique Nash equilibrium.} \]

APPENDIX B: POTENTIAL OF THE MULTICOMMODITY AGGREGATIVE GAME

We now consider the case where the synchronous game has an exact potential function (again, which can be used to show existence of and convergence to a Nash equilibrium [19], [25], [11]). First define “better response” iterative, synchronous existence of and convergence to a Nash equilibrium [19], [25], \( \forall \)

\( x \)

that cannot be canceled by the other two terms on the right-hand-side), and this for an arbitrary \( j \neq i \); hence it must be constant (mapping \( \mathbb{R}^K \to \mathbb{R} \)), \( m_i := \nabla x_i^* (s_{-i}) \). Moreover, \( \partial_j B_i(x_{-i}) + x_j \) must be constant in \( x_j \). Thus, for all \( x, i \neq j \):

\[
\partial_j \Phi(x) = -x_j + x_i d_j(m_i) + b_i, j(x_{-i}, -j),
\]

for some \( b_{i,j}(x_{-i}, -j) \) that does not depend on \( x_i \) or \( x_j \). Since \( \partial_j \partial_i \Phi = \partial_i \partial_j \Phi, d_j(m_i) = d_j(m_j) =: \kappa \neq i, j \) for all \( i \neq j \).

Thus,

\[
\Phi(x) = -\frac{x_j^2}{2} - \frac{x_i^2}{2} + \kappa_i x_i x_j + \beta_j x_j + \beta_i x_i + f_{i,j}(x_{-i}, -j),
\]

for some \( f_{i,j}(x_{-i}, -j) \) and constants \( \beta \). So, we arrive at (15) by induction.

Recall tenant criterion (6) with valuation function \( v_i \) increasing and non-convex and cost functions \( p_i(s) := s \beta_i(s) \) increasing and non-concave.

**Corollary 1.** If costs are quadratic [11], valuations are linear [8], and

\[
\sum_k d_{i,k}^2 \pi_k
\]

is a constant function of \( i \),

(16)

then the game based on the utilities (6) has an exact potential.

This corollary can be directly verified by solving \( \partial_i u_i = 0 \) for \( x_i \) and comparing the result to \( \partial_i \Phi + x_i \) using (15). Condition (16) is needed to achieve \( \kappa_{i,j} = \kappa_{j,i} \) for all \( j \neq i \).