# Peer-to-peer caching systems with selfish peers: A spectral approach

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***Abstract*—We consider a dynamic distributed caching system consisting of an "authoritative" server dispensing content only if the content fails to be found by searching an unstructured peer-to-peer system. In our model, the peers may not be fully cooperative, in probabilistic fashion, in this search process. We use a spectral approach to analyze the performance of this system, specifically the expected number of hops for successfully resolved queries by the peer-to-peer system.**

## I. Introduction

In a simplified peer-to-peer (p2p) search scenario, peers[1] and known service/data objects are given geospatial coordinates. Queries to (presumed) known coordinates are then forwarded to neighboring peers that are closest. Under certain topological conditions (e.g., the presence of long distance neighbors [11]), it can be shown that forwarding is efficient. Typically based on consistent hashing, distributed hash table (DHT) coordinate systems also have good forwarding-delay properties, e.g., [14]; peers are expected to be able to resolve queries for data/service objects which are proximal to themselves in (hash) key space. Such coordinates systems may require name resolution processes to resolve the identity of the peer or object to a key; this can be made complex if the query is an anycast only specifying certain desired attributes. Load balancing (congestion control) in these systems may have to do with how keys are proactively assigned and reactively reassigned to peers and data/service objects, e.g., [10]. Load balancing can be implemented in a centralized, hierarchically decentralized (exploiting superpeers [12]), or fully decentralized, subject to the typical trade-offs. Moreover, peers may need to be appropriately incented to forward queries of others, e.g., [6].

In an unstructured search, as considered in this paper, the contact/resolution time in hops of a single-threaded or limited-scope flooded query (including an anycast) has been studied using techniques from the spectral theory of Markov chains and random graphs, e.g., [3], [4], [1], [8], [5]. Because issues of name resolution and scalability are obviously also in play in an unstructured setting, peers may need to learn from past experience the best way to forward certain classes of queries, i.e., to intelligently "bias" their forwarding decisions. For example, peers may cache the outcomes of past queries as informed by reverse-path forwarding, e.g., [13], and thus correlate classes of queries with neighbors who can best resolve them. Such classification and learning operations may be to costly for a typical peer, possibly even too costly for altruistic and relatively resource-rich superpeers.

In this paper, we consider a hybrid unstructured p2p and client-server framework whose scalability was studied in [8] assuming all peers fully cooperate. That is, they studied how the traffic load imposed on the server and the load on individual peers grow as the number of peers in the peer-to-peer system increases. In their model, the unstructured peer-to-per system is supposed to alleviate the traffic load at the server, i.e., the p2p network dynamically caches the server's content. Though it was shown in [8] that hybrid peer-to-peer system can alleviate the load of a server significantly, it is herein shown how selfish peers degrade the performance of such a system.

This paper is organized as follows. In Section II, the set-up of the hybrid p2p-server caching system is described together with some technical preliminaries. In Section III, probabilistically selfish behavior is studied wherein selfish peers only sometimes act selfishly when handling queries (in to avoid detection as noncooperative peers). A numerical study on a conjecture of Section III is given in Section III-C. Finally, we make some concluding remarks in Section IV.

## II. Problem formulation

Consider a hybrid peer-to-peer system consisting of a server and many peers. The peers of the peer-to-peer (p2p) system form an unstructured network with each peer connected to the server. In [8], the authors analyzed the query load imposed on the server, the load imposed on peers, and the query response time (in forwarding "hops") under two query propagation mechanisms: the random walk and the expanding ring. They showed that for expander graphs, the peers in a hybrid p2p network can significantly alleviate the load of the server: the query load at the server and the query load of the p2p network are bounded as the number of peers increases. For the purposes of comparison, we consider the same model that is used in [8] except that some peers in the system may be selfish while others are altruistic (cooperative).

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[1]In the literature and herein, peers will also be called users, nodes, or vertices (in the peer connectivity graph).

### A. Preliminaries

On a given graph $G(V,E)$ with vertex/node set $V$ corresponding to peers and edge set $E$ corresponding to direct (one "hop") lines of communication between them, a query forwarding process can be modeled as a discrete-time (forwarding hops) random walk. Herein, the random walk is taken to be a Markov chain on $G(V,E)$ with "uniform" transition-probability matrix $R$ whose entries are, for all peers $i,j \in V$,

$$R_{ij} = \begin{cases} \frac{1}{d_i}, & \text{if } i \neq j,\ (i,j) \in E, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $d_i$ is the degree of peer $i$, i.e., the number of edges of peer $i$. Note that the graph $G(V,E)$ uniquely defines such a discrete-time Markov chain.

The steady-state probability for this random walk[2] is

$$\pi_i = \frac{d_i}{\sum_{i=1}^{N} d_i} \quad \forall i \in V,$$

where $N = |V|$ is the number of peers.

A Markov chain with transition-probability matrix $R$ is (time) reversible if $\pi_i R_{ij} = \pi_j R_{ji}$ for all peers $i,j \in V$. Assuming bidirectional edges (i.e., $\forall i,j \in V$, $(i,j) \in E \quad \Leftrightarrow \quad (j,i) \in E$), the matrix given by (1) and any lazy version are both reversible. For a reversible discrete-time Markov chain on a state space of size $N = |V|$ with transition-probability matrix $R$, we index the eigenvalues of $R$ in decreasing order

$$1 = \lambda_1^{(R)} > \lambda_2^{(R)} \geq ... \geq \lambda_N^{(R)} \geq -1.$$

The spectral gap of this reversible Markov chain is $1 - \lambda_2^{(R)}$ (as a result of time reversibility, the Markov rate matrix can be expressed as a self-adjoint operator having a Courant-Fischer spectral representation). The relaxation time of a reversible Markov chain (roughly the time to steady state) is (see equ. (39) of [1][3]):

$$\tau^{(R)} \quad := \quad (1 - \lambda_2^{(R)})^{-1}.$$

### B. Hybrid p2p/client-server model of [8]

The model in [8] consists of a single server and $N$ peers with maximum degree $d := \max_{i \in V} d_i$. It is assumed that the p2p network is connected.

Peers may enter and leave the system. But as soon as a peer departs the system, a new peer enters the system and occupies the exact peer in $V$ which was occupied by the departing peer (this modeling assumption for peer churn is commonly made, also see, e.g., Sec. 2.4 of [2]). As a result, neither the total number of peers in the peer-to-peer system nor its graphical topology changes. Peers stay in the system for independent and identically exponentially distributed time with mean $1/\mu$. To further facilitate analysis, [8] considered the case in which queries are generated for only one item and query requests are generated only by newly entering peers. Each newly entering peer generates a query request with probability $p$ and hence will possess the content under consideration to help resolve subsequent queries. By Little's formula [15], the mean rate at which peers "arrive" is $N/(1/\mu)$, and so the mean rate at which queries are generated is

$$pN\mu. \tag{2}$$

A peer who issues a query first sends to her neighboring peers over the peer-to-peer system. She waits for a response from the peers for $T_{\max}$ hops. Here, we assume that the timeout of the querier is calibrated to the maximum time-to-live (TTL, query timeout) hop-count of the p2p network. If she does not get a response from the peer-to-peer system by $T_{\max}$, she deems the process of searching the p2p network as having failed and sends a new request directly to the server.

Two query propagation mechanisms are studied in [8]: the random walk and the expanding ring. This paper considers only the random walk mechanism, where a peer issuing a query sends a query packet to one of her neighbors which is chosen at random. A peer that receives the query packet checks whether it has the requested item. If she has the item, she replies to and shares the item with the peer who initiated the query. Otherwise, she forwards the packet to one of its neighbors also chosen at random. Again, to simplify analysis, it is assumed that peers can receive the same query packet multiple times (i.e., no "taboo" list is maintained on the packet to avoid cycles or enable reverse-path forwarding of a successfully resolved query). Under these circumstances, it makes sense that each query packet has an expiration time corresponding to $T_{\max}$, i.e., again a TTL counter (in hops) is decremented on the packet and a peer does not forward a packet with TTL$= 0$.

The transmission (relay) time of a query packet is assumed geometrically distributed with mean $\alpha/(1-\alpha)$, i.e., the holding time of lazy random walk with transition probability matrix $L = (1-\alpha)R + \alpha I$ for $\alpha \in (0,1)$. In the following, excessive holding times and repeated visits by a query to the same peer may model how a selfish or malicious relaying peer depletes a query's TTL.

The p2p graph under consideration may in fact represent a peer-to-peer *overlay* network, where one hop in the overlay may correspond to several "physical" hops. Considering the resulting variability in communication delay and variability in forwarding delay within each peer, we may take the discrete-time Markov chain with transition-probability matrix $L$ or $R$ as "embedded" within a continuous-time Markov chain by associating independent (continuously) exponentially distributed random variables with mean $1/\delta_i$ to each peer $i$. Hence, from $L$ a transition-rate matrix $Q$ of a continuous-time Markov chain:

$$Q_{ij} \quad = \quad \frac{\delta_i}{d_i} \qquad \text{if } i \neq j, (i,j) \in E.$$

That is, the holding times peer $i$ are exponentially distributed with mean $1/\delta_i$. This continuous-time Markov chain model of

[2] Or for a "lazy" version with transition rate matrix $L = (1-\alpha)R + \alpha I$ (irrespective of $\alpha \in (0,1)$).

[3] $\lambda$ is an eigenvalue of $R$ (associated to eigenvectors $\underline{v}$) if and only if $\alpha + (1-\alpha)\lambda$ is an eigenvalue of $L$ (associated with the same group of eigenvectors). Hence, the relaxation time of the lazy random walk is $\tau^{(L)} = (1-\lambda_2^{(L)})^{-1} = ((1-\alpha)(1-\lambda_2^{(R)}))^{-1} = \tau^{(R)}(1-\alpha)$, where $1-\lambda_2^{(L)}$ is the spectral gap of $L$.

query forwarding is also time reversible with invariant

$$\pi_i \quad \propto \quad d_i/\delta_i.$$

Of course, using a continuous-time Markov model, the dimension of all time quantities can be taken to be seconds (instead of hops). Note that the results of [8] employ bounds in Chapter 3 of [1] for continuous-time Markov chains.

Also in [8], the dynamics according to which peers enter and leave the system and the query dynamics in which a query propagates over the p2p network are decoupled by assuming that the total query response time $T_{\max}\alpha/(1-\alpha)$ is negligible compared to the lifetime of a peer in the system ($1/\mu$). Therefore, the system's querying dynamics work on a static p2p graph, i.e., it's assumed that no peer leaves or enters the system while a query propagates among them.

### C. *A new model with selfish peers*

In the random walk model proposed in [8], all peers are cooperative in the sense that either all peers forward a query request if they do not have the content, or they share it with the peer who issued the query if they have the content. Clearly if all peers are selfish, then the hybrid peer-to-peer system is identical to a centralized system with a single central server. Hence, the load on the server obviously would not be bounded, even in expander networks, as $N \to \infty$.

We assume that selfish peers and altruistic peers coexist in the peer-to-peer system. Say a fraction $\sigma$ of the peers are selfish while the rest are cooperative peers.

When a peer departs the p2p network and a new peer arrives replacing it, the new peer does not necessarily inherit the selfish property of the departing one: it is possible that the exiting peer is selfish and the arriving peer is cooperative or vice versa. It is also assumed that the new entering peer is selfish with probability $\sigma$. The types of selfish behavior include:

- (S1) An **impatient** peer does not first send a query request to the p2p network, but instead directly accesses the server to get the item without any delay.
- (S2) A **non-resolving** peer does not share the requested content even though the peer possesses it; instead it merely forwards/relays the query.
- (S3) A **non-forwarding** peer does not forward queries and does not possess the queried content.
- (S4) A **blackhole** peer conserves their resources by neither forwarding a query nor resolving a query even though the peer possesses the queried content, i.e., possessing content and being a blackhole are "independent" properties of a peer.
- (S5) A **completely selfish** peer is an impatient blackhole.

In [9] we conducted a study of "statically" selfish behavior wherein a selfish peer behaves in the same noncooperative fashion at every opportunity to forward a query.

## III. Probabilistically selfish behavior

In this section, we consider a more dynamic type of selfish behavior wherein an arriving peer is selfish with probability $\sigma$ and will thereafter behave selfishly when handling queries only with probability $\beta$, but here a selfish peer may behave differently when handling the same query more than once[4].

For brevity, we provide the details for probabilistic blackhole peers only. In [9], we studied probabilistic selfishness using simple "coupling" arguments[5] and thus leading to a lower bound on the expected number of hops of a successfully resolved query by the p2p network (cf., first inequality of (10)). We will herein use a "spectral" approach to derive an upper bound on the same quantity. One can also consider selfish peers that change their behavior from query to query, i.e., may "statically" act cooperatively for one query and act selfishly for another.

In certain special cases, probabilistically selfish peers are easy to analyze; note that the distribution of the stopping time of the random walk is simply geometrically distributed with parameter $\beta$, in a manner independent of the graphical properties of the p2p network, when $\sigma = 1 = p$. Though this approach may lead to numerically approximate answers, to more precisely study performance for a more general case, we need to revisit the fundamental results employed by [8].

### A. *Background on spectral properties of random walks*

The smallest strictly positive eigenvalue (or spectral gap)[6] of the transition rate matrix $-Q = [-Q_{ij}]$ is[7]

$$\begin{aligned} \lambda &= \inf_{g:V\to\mathbb{R}} \frac{\mathcal{E}(g,g)}{\mathsf{var}_\pi(g)} \\ &= \inf_{g:V\to\mathbb{R},\ \mathsf{E}_\pi g=0} \frac{\mathcal{E}(g,g)}{\mathsf{E}_\pi g^2}, \end{aligned}$$

where the Dirichlet form is

$$\mathcal{E}(g,g) \quad := \quad \frac{1}{2}\sum_{i\in V}\sum_{j\in V, j\neq i} \pi_i Q_{ij}(g_i - g_j)^2,$$

and the variance and second moment are

$$\begin{aligned} \mathsf{var}_\pi(g) &:= \sum_{i\in V}\pi_i \left(g_i - \sum_{j\in V}\pi_j g_j\right)^2, \\ \mathsf{E}_\pi(g) &:= \sum_{i\in V}\pi_i g_i^2. \end{aligned}$$

Now consider a *fixed* contact set $A \subset V$ terminating the random walk $Q$ before timeout (p. 40-41 of [1]), i.e., the substochastic $|N-|A|| \times |N-|A||$ transition rate matrix $Q^{(A)}$ restricted to $V\backslash A = A^c$ with entries

$$Q_{ij}^{(A)} \quad = \quad \begin{cases} Q_{ij} & \text{if } i,j \in A^c \\ 0 & \text{else} \end{cases}$$

[4]Recall that our random walk is assumed memoryless. Note that this is another way to model how a node may (selfishly or maliciously) attempt deplete a query's TTL.

[5]Relating to the performance under static selfishness with the same network parameters, and which, in turn, is related to a fully cooperative system with modified querying parameters $p$, e.g., $p \to p(1-\sigma)$.

[6]By eqn. (38) of [1], the the relaxation time of the continuous-time random walk with time-invariant transition-rate matrix $Q$ on $V$ is $\tau = 1/\lambda$.

[7]The zero eigenvalue has left eigenvector which is the unique invariant $\underline{\pi}$ of the assumed irreducible, time-reversible $Q$, and a right eigenvector with all entries equal to 1.

By (69) of [1],

$$\mathsf{E}_{\pi_{A^c}} T_A \;=\; \frac{E_{\pi T_A}}{\pi(A^c)} \;\leq\; \frac{1}{\lambda_A} \tag{3}$$

where (equ. (84) of [1])

$$\lambda_A \;=\; \inf_{g_A \equiv 0} \frac{\mathcal{E}(g,g)}{\mathsf{var}_\pi(g)} \;\geq\; \lambda,$$

because the expression for $\lambda$ is not restricted to infimum over mappings $g : V \to \mathbb{R}$ such that $g_i = 0 \;\forall i \in A$.

Note that in our setting, the initial distribution for search is the uniform $u_{A^c}$ on $A^c$ not the invariant $\pi_{A^c}$ of $Q^{(A)}$. As (8) of [8], we can relate

$$\begin{aligned}
&\mathsf{E}_{\pi_{A^c}} \min_{i \in A^c} \frac{u_{A^c,i}}{\pi_{A^c,i}}\; T_A \leq \mathsf{E}_{u_{A^c}} T_A \leq \mathsf{E}_{\pi_{A^c}} \max_{i \in A^c} \frac{u_{A^c,i}}{\pi_{A^c,i}}\; T_A, \\
&\Rightarrow\; \mathsf{E}_{\pi_{A^c}} \min_{i \in A^c} \frac{\sum_{j \in A^c} d_j/\delta_j}{|A^c| d_i/\delta_i}\; T_A \leq \mathsf{E}_{u_{A^c}} T_A \\
&\qquad \leq \mathsf{E}_{\pi_{A^c}} \max_{i \in A^c} \frac{\sum_{j \in A^c} d_j/\delta_j}{|A^c| d_i/\delta_i}\; T_A, \\
&\Rightarrow \mathsf{E}_{\pi_{A^c}} \frac{\min_{i \in A^c} d_j/\delta_j}{\max_{i \in A^c} d_j/\delta_j}\; T_A \leq \mathsf{E}_{u_{A^c}} T_A \\
&\qquad \leq \frac{\max_{i \in A^c} d_j/\delta_j}{\min_{i \in A^c} d_j/\delta_j} \mathsf{E}_{\pi_{A^c}} T_A.
\end{aligned} \tag{4}$$

By (38) of [1], a sequence of graphs $\{G(N)\}_{N=2}^{\infty}$, is an expander family if

$$\frac{1}{\hat{\lambda}} \;:=\; \limsup_{N \to \infty} \frac{1}{\lambda(N)} \;<\; \infty,$$

i.e., $\hat{\lambda} > 0$ (see also [7] and (3) of [8]). Note that for all $A(N) \subset V(N)$,

$$\limsup_{N \to \infty} \frac{1}{\lambda_{A(N)}(N)} \;\leq\; \frac{1}{\hat{\lambda}}$$

since $\lambda_A(N) \geq \lambda(N) \;\forall A \subset V(N)$. So by (3) above, for an expander family of graphs $G(N)$ and any sequence of contact sets $A(N) \subset V(N)$,

$$\limsup_{N \to \infty} \mathsf{E}_{\pi_{A^c(N)}} T_{A(N)} \;\leq\; \frac{1}{\hat{\lambda}} \;<\; \infty. \tag{5}$$

### B. Probabilistic blackhole peers

Probabilistic selfish behavior will effectively impact the transition-rate matrix (or transition-probability matrix in discrete time (hops)). Now let $\sigma$ represent the probability that a peer is a probabilistic blackhole and let $\beta$ be the probability that a blackhole peer does not forward when queried. *Given* that a peer $i$ is a probabilistic blackhole, i.e., $i \in A^{\mathsf{PBH}}$, its discrete-time transition probabilities to its $d_i$ neighbors $j$ are now changed to

$$\tilde{Q}_{ij} \;=\; (1-\beta)\delta_i/d_i \;\text{ if } i \neq j,\; (i,j) \in E.$$

Given $A^{\mathsf{PBH}}$, we see that $\tilde{Q}$ is time-reversible with invariant

$$\tilde{\pi}_i \;\propto\; \begin{cases} \frac{d_i}{\delta_i(1-\beta)} & \text{if } i \in A^{\mathsf{PBH}} \\ \frac{d_i}{\delta_i} & \text{else} \end{cases}$$

That is, the tilde ($\sim$) indicates $\beta > 0$.

Note that that if we take $\beta = 1$, then we return to the case of "static" blackholes of [9].

In the following, fix $V$ (and $N := |V|$). Conditioning on $A^{\mathsf{PBH}}$ (which is binomial$(N-1, \sigma)$ in size) and the independent contact set $A$ (which is binomial$(N-1, p)$ in size), we get for the $\tilde{Q}$ dynamics on $A^c$:

$$\begin{aligned}
\mathsf{E}_{\tilde{\pi}_{A^c}} \tilde{T}_A = \sum_{A^{\mathsf{PBH}} \subset V} \sum_{A \subset V} \mathsf{E}_{\tilde{\pi}_{A^c}} (\tilde{T}_A \mid A^{\mathsf{PBH}}, A) p^{|A|} (1-p)^{N-1-|A|} \\
\cdot\, \sigma^{|A^{\mathsf{PBH}}|} (1-\sigma)^{N-1-|A^{\mathsf{PBH}}|},
\end{aligned} \tag{6}$$

where we use $N-1$ instead of $N$ to account for the querying peer.

Therefore, in the following, assume $A^{\mathsf{PBH}}$ and $A$ are *given* and define

$$B := A^{\mathsf{PBH}} \backslash A \quad \text{and} \quad C := A^c \backslash A^{\mathsf{PBH}},$$

i.e., the partition $A^c = C \cup B$. Now, by (69) of [1] (see also (83)-(87) and Corollary 34),

$$\mathsf{E}_{\tilde{\pi}_{A^c}} (\tilde{T}_A \mid A^{\mathsf{PBH}}, A) \;\leq\; \frac{1}{\tilde{\lambda}_A}, \tag{7}$$

where ((84) of [1])

$$\tilde{\lambda}_A \;=\; \inf_{\tilde{g}:\; \mathsf{E}_{\tilde{\pi}_{A^c}} \tilde{g} = 0,\; g_A \equiv 0,\; g \not\equiv 0} \frac{\frac{1}{2} \sum_{i,j \in A^c} \tilde{\pi}_i \tilde{Q}_{ij} (\tilde{g}_i - \tilde{g}_j)^2}{\mathsf{E}_{\tilde{\pi}_{A^c}} \tilde{g}^2}.$$

Note that we have suppressed indication of the dependence on $A^{\mathsf{PBH}}$ of $\tilde{\lambda}_A$. Let "$\tilde{g} \not\equiv \tilde{g}_B$" represent that $\tilde{g}$ is not constant ($\tilde{g}_B$) on $B$. Also let

$$\begin{aligned}
\tilde{\gamma}_A^{(\mathrm{C+B})} \;&=\; \inf_{\tilde{g}:\; \mathsf{E}_{\tilde{\pi}_{A^c}} \tilde{g} = 0;\; \tilde{g} \not\equiv \tilde{g}_B \text{ or } \tilde{g} \not\equiv \tilde{g}_C} \\
&\qquad \frac{\frac{1}{2} \sum_{(i,j) \in B \times B \text{ or } C \times C} \tilde{\pi}_i \tilde{Q}_{ij} (\tilde{g}_i - \tilde{g}_j)^2}{\mathsf{E}_{\tilde{\pi}_{A^c}} \tilde{g}^2}, \\
\tilde{\gamma}_A^{(\mathrm{CB})} \;&=\; \inf_{\tilde{g}_B, \tilde{g}_C \neq 0 \;:\; \tilde{\pi}_C \tilde{g}_C + \tilde{\pi}_B \tilde{g}_B = 0} \\
&\qquad \frac{\frac{1}{2} (\tilde{g}_C - \tilde{g}_B)^2 \sum_{i \in C, j \in B} \tilde{\pi}_i \tilde{Q}_{i,j}}{\tilde{\pi}_C \tilde{g}_C^2 + \tilde{\pi}_B \tilde{g}_B^2} \\
&>\; 0 \;\text{ since } \mathrm{sgn}(g_C) \neq \mathrm{sgn}(g_B), \\
\tilde{\gamma}_A \;&=\; \begin{cases} \min\{\tilde{\gamma}_A^{(\mathrm{C+B})}, \tilde{\gamma}_A^{(\mathrm{CB})}\} & \text{if } \tilde{\gamma}_A^{(\mathrm{C+B})} > 0 \\ \tilde{\gamma}_A^{(\mathrm{CB})} & \text{else} \end{cases}
\end{aligned}$$

Clearly,

$$\tilde{\lambda}_A \geq \tilde{\gamma}_A > 0 \quad \text{and} \quad \lambda_A \geq \gamma_A > 0. \tag{8}$$

We will show how the probabilistic black hole peers affect the bounds on mean contact time given by the lower bound $\tilde{\gamma}_A$ on the spectral gap $\tilde{\lambda}_A$.

*Theorem 1:*

$$\frac{\gamma_A}{1-\beta} \;\geq\; \tilde{\gamma}_A \;\geq\; (1-\beta)^2 \gamma_A.$$

*Proof:* Since $\tilde{\pi}_j = \pi_j/(1-\beta)$ on $B$, otherwise $\tilde{\pi}_i = \pi_j$ on $C$, there is a one-to-one mapping between eligible functions $\tilde{g}$ when computing $\tilde{\gamma}_A$ and those eligible functions $g$ for $\gamma_A$

without probabilistic black hole peers (i.e., when $\sigma = 0$ or equivalently $\beta = 0$):

$$\tilde{g}_i = \begin{cases} g_i(1-\beta) & \text{for } i \in B \\ g_i & \text{for } i \in C \\ 0 = g_i & \text{for } i \in A \end{cases}$$

Therefore, we can write

$$\tilde{\gamma}_A^{(\mathrm{C+B})} = \frac{1}{2} \inf_{g_A \equiv 0,\ g \not\equiv 0,\ \mathsf{E}_{\pi_{A^c}} g = 0} \frac{\sum_{i,j\in C} \pi_i Q_{ij}(g_i - g_j)^2 + (1-\beta)^2 \sum_{i,j\in B} \pi_i Q_{ij}(g_i-g_j)^2}{\sum_{i\in C} \pi_i g_i^2 + (1-\beta)\sum_{i\in B}\pi_i g_i^2}.$$

Thus,

$$\gamma_A^{(\mathrm{C+B})}(1-\beta)^2 \ \leq\ \tilde{\gamma}_A^{(\mathrm{C+B})} \ \leq\ \frac{\gamma_A^{(\mathrm{C+B})}}{1-\beta}, \qquad (9)$$

where

$$\gamma_A^{(\mathrm{C+B})} = \inf_{g:\ g_A \equiv 0,\ g\not\equiv 0,\ \mathsf{E}_{\pi_{A^c}} g = 0} \frac{\sum_{(i,j)\in B\times B \text{ or } C\times C} \pi_i Q_{ij}(g_i-g_j)^2}{\mathsf{E}_{\pi_{A^c}} g^2}.$$

For $\tilde{\gamma}_A^{(\mathrm{CB})}$, again note that eligible constants $g_C$ and $g_B$ must be of opposite sign so that $\mathsf{E}_{\pi_{A^c}} g = 0$. Thus,

$$\tilde{\gamma}_A^{(\mathrm{CB})} = \inf_{\tilde{g}_B, \tilde{g}_C \neq 0\ :\ \tilde{\pi}_C \tilde{g}_C + \tilde{\pi}_B \tilde{g}_B = 0} \frac{\frac{1}{2}(g_C - (1-\beta)g_B)^2 \sum_{i\in C, j\in B} \pi_i Q_{i,j}}{\sum_{i\in C}\pi_i g_i^2 + (1-\beta)\sum_{i\in B}\pi_i g_i^2}$$

will also satisfy a bound like (9); and consequently so will $\tilde{\gamma}_A$ by its definition. □

By this claim, (6), (7) and (8), we get that

$$\mathsf{E}_{\pi_{A^c}}(T_A \mid A^{\mathsf{PBH}}, A) \leq \mathsf{E}_{\tilde{\pi}_{A^c}}(\tilde{T}_A \mid A^{\mathsf{PBH}}, A) \qquad (10)$$
$$\leq \frac{1}{(1-\beta)^2} \sum_{A\subset V} \frac{1}{\gamma_A} p^{|A|}(1-p)^{N-|A|} \sigma^{|A^{\mathsf{PBH}}|}(1-\sigma)^{N-|A^{\mathsf{PBH}}|}$$

where the first inequality is immediate by coupling and we note that $\gamma_A$ (like $\tilde{\gamma}_A$, of course, but unlike $\lambda_A$) depends on $A^{\mathsf{PBH}}$ through $B$ and $C$.

### C. Numerical example

Consider (7), (8) and Theorem 1. Recalling how we can relate to $\mathsf{E}_{u_{A^c}} \tilde{T}_A$ through (4), we now numerically explore the conjecture that

$$\mathsf{E}_{\pi_{A^c}} T_A \ \leq\ \mathsf{E}_{\tilde{\pi}_{A^c}} \tilde{T}_A \ \leq\ \mathsf{E}_{\pi_{A^c}} T_A/(1-\beta)^2. \qquad (11)$$

That is, we want to test the postulated second inequality motivated by the second inequality of Claim 1. Figure 1 verifies this conjecture for the case where TTL$\equiv T_{\max} = 100$, $p = 0.2$, and $N = 200$, and different $\sigma = \beta$ values.

## IV. Concluding remarks

Note how time reversibility is important in order to be able to apply the spectral results of [1] Chapter 3. Also note that we have modeled how a probabilistic blackhole randomly depletes the TTL of the query packet by repeatedly forwarding the query to itself. For our model of a p2p caching system with selfish peers, we verified numerically a conjecture (11) based on our main result (10) regarding the expected number of hops of successfully resolved queries.

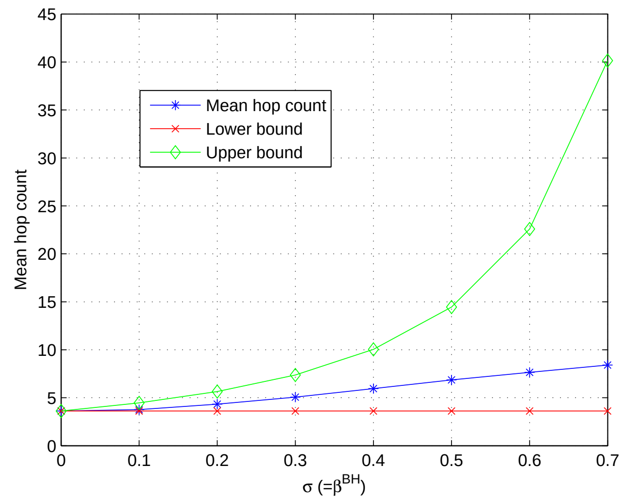


Fig. 1. Verification of conjecture (11)

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